

## FOURIER SERIES INVOLVING THE $\bar{H}$ FUNCTION

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### Abstract

The present paper deals with the use of an integral having the product of sine function, exponential function, Kampé de Fériet functions and  $\bar{H}$ -function as integrand to evaluate three specific Fourier series. A multiple integral involving the  $\bar{H}$ -function also been evaluated and its use has been shown to derive a multiple exponential Fourier series. In the end some interesting particular cases have also been discussed.

**Keywords:** Fourier series,  $\bar{H}$ -function, Kampé de Fériet function.  
 2000 AMS Subject Classification: 33C60.

### INTRODUCTION

The  $\bar{H}$ -function occurring in this paper will be defined and represented in the following manner [Buschman and Srivastava 1990]:

$$\begin{aligned} \bar{H}_{P,Q}^{M,N}[z] &= \bar{H}_{P,Q}^{M,N} \left[ z \left| \begin{array}{l} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{array} \right. \right] \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \bar{\phi}(\xi) z^\xi d\xi \end{aligned} \quad (1.1)$$

where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^M \Gamma(b_j - \beta_j \xi) \prod_{j=1}^N \{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}}{\prod_{j=M+1}^Q \{\Gamma(1 - b_j + \beta_j \xi)\}^{B_j} \prod_{j=N+1}^P \Gamma(a_j - \alpha_j \xi)} \quad (1.2)$$

which contains fractional powers of some of the gamma functions. Here, and throughout the paper  $a_j (j = 1, \dots, P)$ ,  $b_j (j = 1, \dots, Q)$  are complex parameters,  $\alpha_j \geq 0 (j = 1, \dots, P)$  and  $\beta_j \geq 0 (j = 1, \dots, Q)$  (not all zero simultaneously) and the exponents  $A_j (j = 1, \dots, N)$  and  $B_j (j = M + 1, \dots, Q)$  can take on non-integer values.

Buschman and Srivastava [1990] has proved that the integral on the right hand side of Eq. (1.1) is absolutely convergent when  $\Omega > 0$  and  $|\arg z| < \frac{1}{2} \pi \Omega$ , where

$$\Omega \equiv \sum_{j=1}^M \beta_j + \sum_{j=1}^N A_j \alpha_j - \sum_{j=M+1}^Q \beta_j B_j - \sum_{j=N+1}^P \alpha_j > 0. \quad (1.3)$$

For further details of  $\bar{H}$ -function one can refer the original paper of Buschman and Srivastava [1990].

Kampé de Fériet hypergeometric function will be represented as follows:

$$F \left( \begin{matrix} p \\ \mu \\ q \\ \sigma \end{matrix} \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, b'_1, \dots, b_\mu, b'_\mu \\ c_1, \dots, c_q \\ d_1, d'_1, \dots, d_\sigma, d'_\sigma \end{matrix} \middle| xy \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{m+n} \prod_{j=1}^{\mu} \{(b_j)_m (b'_j)_n\}}{\prod_{j=M+1}^q (c_j)_{m+n} \prod_{j=1}^{\sigma} \{(d_j)_m (d'_j)_n\}} \frac{x^m y^n}{m!n!} \quad (1.4)$$

$(p + \nu < q + \sigma + 1$  or  $p + \nu = q + \sigma + 1$  and  $|x| + |y| < \min(1, 2^{q-p+1})$ );

$$= -\frac{1}{4\pi^2 K} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \psi(s, t) \Gamma(-s) \Gamma(-t) (-x)^s (-y)^t ds dt$$

where

$$K = \frac{\prod_{j=1}^p \Gamma(a_j) \prod_{j=1}^{\mu} \{\Gamma(b_j) \Gamma(b'_j)\}}{\prod_{j=M+1}^q \Gamma(c_j) \prod_{j=1}^{\sigma} \{\Gamma(d_j) \Gamma(d'_j)\}} \quad (1.5)$$

and

$$\psi(s, t) = \frac{\prod_{j=1}^p \Gamma(a_j + s + t) \prod_{j=1}^{\mu} \{\Gamma(b_j + s) \Gamma(b'_j + t)\}}{\prod_{j=M+1}^q \Gamma(c_j + s + t) \prod_{j=1}^{\sigma} \{\Gamma(d_j + s) \Gamma(d'_j + t)\}} \quad (1.6)$$

By substituting  $\mu = 0 = \sigma$ , it changes in the following form:

$$F \left( \begin{matrix} p \\ \mu \\ q \\ \sigma \end{matrix} \middle| \begin{matrix} a_1, \dots, a_p \\ \text{---} \\ c_1, \dots, c_q \\ \text{---} \end{matrix} \middle| xy \right) = {}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ c_1, \dots, c_q \end{matrix} ; x + y \right) \quad (1.7)$$

For further detail one can refer the monography by Appel and Kampé de Fériet [1926].

Mishra [1990] has evaluated

$$\int_0^\pi (\sin x)^{w-1} e^{imx} {}_pF_q \left[ \begin{matrix} \alpha_p; \\ \beta_q; \end{matrix} C(\sin x)^{2h} \right] dx = \frac{\pi e^{im\pi/2}}{2^{w-1}} \sum_{r=0}^\infty \frac{(\alpha_p)_r C^r \Gamma(w+2hr)}{(\beta_q)_r r! 4^{hr} \Gamma(\frac{w+2hr \pm M+1}{2})} \quad (1.8)$$

Where  $(\alpha)_p$  denotes  $\alpha_1, \dots, \alpha_p$ ;  $\Gamma(a \pm b)$  represents  $\Gamma(a+b), \Gamma(a-b)$ ;  $h$  is a positive integer;  $p < q$  and  $\text{Re}(w) > 0$ . Recall the following elementary integrals:

$$\int_0^\pi e^{i(m-n)x} dx = \begin{cases} \pi, & m = n; \\ 0, & m \neq n; \end{cases} \quad (1.9)$$

$$\int_0^\pi e^{imx} \cos nx dx = \begin{cases} \frac{\pi}{2}, & m = n \neq 0; \\ \pi, & m = n = 0; \\ 0, & m \neq n; \end{cases} \quad (1.10)$$

$$\int_0^\pi e^{imx} \sin nx dx = \begin{cases} i\frac{\pi}{2}, & m = n; \\ 0, & m \neq n; \end{cases} \quad (1.11)$$

Provided either both  $m$  and  $n$  are odd or both  $m$  and  $n$  are even integers. For brevity, the following notations shall be used;

$$\frac{\prod_{k=1}^E (e_k)_{r+t} \prod_{k=1}^F (f_k)_r \prod_{k=1}^{F'} (f'_k)_t}{\prod_{k=1}^G (g_k)_{r+t} \prod_{k=1}^H (h_k)_r \prod_{k=1}^{H'} (h'_k)_t} = \epsilon$$

$$\frac{\prod_{k_1=1}^{E_1} (e_{1k_1})_{r_1+t_1} \prod_{k_1=1}^{F_1} (f_{1k_1})_{r_1} \prod_{k_1=1}^{F'_1} (f'_{1j_1})_{t_1}}{\prod_{k_1=1}^{G_1} (g_{1k_1})_{r_1+t_1} \prod_{k_1=1}^{H_1} (h_{1k_1})_{r_1} \prod_{k_1=1}^{H'_1} (h'_{1k_1})_{t_1}} = \epsilon_1$$

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$$\frac{\prod_{k_n=1}^{E_n} (e_{nk_n})_{r_n+t_n} \prod_{k_n=1}^{F_n} (f_{nk_n})_{r_n} \prod_{k_n=1}^{F'_n} (f'_{nj_n})_{t_n}}{\prod_{k_n=1}^{G_n} (g_{nk_n})_{r_n+t_n} \prod_{k_n=1}^{H_n} (h_{nk_n})_{r_n} \prod_{k_n=1}^{H'_n} (h'_{nk_n})_{t_n}} = \epsilon_n$$

## INTEGRAL

The integrals to be evaluated are

$$\int_0^\pi (\sin x)^{w-1} e^{imx} F_{G;H;H'}^{E;F;F'} \left[ \begin{matrix} (e); (f); (f'); \\ (g); (h); (h'); \end{matrix} \middle| \begin{matrix} \alpha(\sin x)^{2\rho} \\ \beta(\sin x)^{2\gamma} \end{matrix} \right]$$

$$\times \bar{H}_{M,N}^{P,Q} \left[ z(\sin x)^{2\sigma} \middle| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right] dx = \frac{\sqrt{(\pi)} e^{im\pi/2}}{2^{\omega-1}} \sum_{r,t=0}^{\infty} \epsilon \frac{(\alpha/4^\rho)^r (\beta/4^\gamma)^t}{r! t!}$$

$$\times \bar{H}_{P+1,Q+2}^{M,N+1} \left[ \frac{z}{4^\sigma} \middle| \begin{matrix} (1-\omega-2\rho r-2\gamma t, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1-\omega-2\rho r-2\gamma t \pm m}{2}, \sigma; 1 \right) \end{matrix} \right] \quad (2.1)$$

provided that  $|\arg z| < \frac{1}{2}\pi\Omega$ , and  $Re(w) > 0$ ;  $\alpha, \beta, \rho, \gamma, \sigma, z$  are positive integers  $i = 1, \dots, N, \Omega$  is denoted by Eq.(1.3). Now making an application to Eq. (2.1), a multiple integral can be derived:

$$\int_0^\pi \dots \int_0^\pi (\sin x)^{w_1-1} \dots (\sin x)^{w_n-1} e^{i(m_1 x_1 + \dots + m_n x_n)}$$

$$\times F_{G_1;H_1;H'_1}^{E_1;F_1;F'_1} \left[ \begin{matrix} (e_1); (f_1); (f'_1); \\ (g_1); (h_1); (h'_1); \end{matrix} \middle| \begin{matrix} \alpha_1(\sin x_1)^{2\rho_1} \\ \beta_1(\sin x_1)^{2\gamma_1} \end{matrix} \right] \dots F_{G_n;H_n;H'_n}^{E_n;F_n;F'_n} \left[ \begin{matrix} (e_n); (f_n); (f'_n); \\ (g_n); (h_n); (h'_n); \end{matrix} \middle| \begin{matrix} \alpha_n(\sin x_n)^{2\rho_n} \\ \beta_n(\sin x_n)^{2\gamma_n} \end{matrix} \right]$$

$$\times \bar{H}_{M,N}^{P,Q} \left[ z(\sin x_1)^{2\sigma_1} \dots (\sin x_n)^{2\sigma_n} \right] dx_1 \dots dx_n$$

$$= \frac{(\pi)^n e^{i(m_1 + \dots + m_n)\pi/2}}{2^{(\omega_1 + \dots + \omega_n) - n}} \sum_{r_1, t_1=0}^{\infty} \dots \sum_{r_n, t_n}^{\infty} (\epsilon_1 \dots \epsilon_n) \frac{(\alpha_1/4^{\rho_1})^{r_1} (\beta_1/4^{\gamma_1})^{t_1}}{r_1! t_1!} \dots \frac{(\alpha_n/4^{\rho_n})^{r_n} (\beta_n/4^{\gamma_n})^{t_n}}{r_n! t_n!}$$

$$\times \bar{H}_{P+n, Q+2n}^{M, N+n} \left[ \frac{z}{4^{(\sigma_1 + \dots + \sigma_n)}} \middle| \begin{matrix} (1-\omega_1-2\rho_1 r_1-2\gamma_1 t_1, 2\sigma_1; 1) \dots (1-\omega_n-2\rho_n r_n-2\gamma_n t_n, 2\sigma_n; 1), \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1-\omega_1-2\rho_1 r_1-2\gamma_1 t_1 \pm m_1}{2}, \sigma_1; 1 \right) \right. \\ \left. (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \right. \\ \left. \dots \left( \frac{1-\omega_n-2\rho_n r_n-2\gamma_n t_n \pm m_n}{2}, \sigma_n; 1 \right) \right] \quad (2.2)$$

Provided that all the conditions of Eq. (2.1) are satisfied and  $\text{Re}(w_i) > 0$ ,  $\sigma_i, \alpha_i, \beta_i, \rho_i, \gamma_i, z_i$  are positive integers ( $i = 1, \dots, n$ ).

### PROOF

To prove Eq. (2.1) expand the  $\overline{H}$ -function into the Mellin-Barnes type integral. Now on changing the order of integration, which is permissible under the conditions stated with the integral, the integral readily follows from Eq. (1.8).

On applying the same procedure as above the integral Eq. (2.2) can be derived easily.

## EXPONENTIAL FOURIER SERIES

Let

$$f(x) = (\sin x)^{w-1} F_{G;H;H'}^{E;F;F'} \left[ \begin{matrix} (e); (f); (f'); \\ (g); (h); (h'); \end{matrix} ; \begin{matrix} \alpha(\sin x)^{2\rho} \\ \beta(\sin x)^{2\gamma} \end{matrix} \right] \\ \times \overline{H}_{M,N}^{P,Q} \left[ z(\sin x)^{2\sigma} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] dx = \sum_{p=-\infty}^{\infty} A_p e^{-ipx} \quad (3.1)$$

which is valid due to  $f(x)$  is continuous and of bounded variation with interval  $(0, \pi)$ . Now, multiplying by  $e^{imx}$  both sides in Eq. (3.1) and integrating it with respect to  $x$  from  $0$  to  $\pi$ , and then making an appeal to Eqs. (1.9) and (2.1), one gets

$$A_p = \frac{e^{im\pi/2}}{2^{w-1}} \sum_{r,t=0}^{\infty} \epsilon \frac{(\alpha/4^p)^r}{r!} \frac{(\beta/4^\gamma)^t}{t!} \\ \times \overline{H}_{P+1,Q+1}^{M,N+1} \left[ \frac{z}{4^\sigma} \left| \begin{matrix} (1 - \omega - 2\rho r - 2\gamma t, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1 - \omega - 2\rho r - 2\gamma t \pm m}{2}, \sigma; 1 \right) \end{matrix} \right. \right] \quad (3.2)$$

An application to Eqs. (3.1) and (3.2) gives the required exponential Fourier series:

$$(2 \sin x)^{w-1} F_{G;H;H'}^{E;F;F'} \left[ \begin{matrix} (e); (f); (f'); \\ (g); (h); (h'); \end{matrix} ; \begin{matrix} \alpha(\sin x)^{2\rho} \\ \beta(\sin x)^{2\gamma} \end{matrix} \right] \\ \times \overline{H}_{M,N}^{P,Q} \left[ z(\sin x)^{2\sigma} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] \\ = \sum_{p=-\infty}^{\infty} \sum_{r,t=0}^{\infty} e^{ip(\pi/2-x)} \epsilon \frac{(\alpha/4^p)^r}{r!} \frac{(\beta/4^\gamma)^t}{t!}$$

$$\times \bar{H}_{P+1, Q+1}^{M, N+1} \left[ \frac{z}{4^\sigma} \left| \begin{array}{l} (1 - \omega - 2\rho r - 2\gamma t, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N+1, P} \\ (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M+1, Q} \left( \frac{1 - \omega - 2\rho r - 2\gamma t \pm m}{2}, \sigma; 1 \right) \end{array} \right. \right]. \quad (3.3)$$

## COSINE FOURIER SERIES

Let

$$f(x) = (\sin x)^{w-1} F_{G; H; H'}^{E; F; F'} \left[ \begin{array}{l} (e); (f); (f'); \quad \alpha (\sin x)^{2\rho} \\ (g); (h); (h'); \quad \beta (\sin x)^{2\gamma} \end{array} \right]$$

$$\times \bar{H}_{M, N}^{P, Q} \left[ z (\sin x)^{2\sigma} \left| \begin{array}{l} (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N+1, P} \\ (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M+1, Q} \end{array} \right. \right] = \frac{B_0}{2} + \sum_{p=1}^{\infty} B_p \cos px \quad (4.1)$$

Integrating both sides with respect to  $x$  from  $0$  to  $\pi$ , one gets

$$\frac{B_0}{2} = \frac{1}{\sqrt{\pi}} \sum_{r, t=0}^{\infty} \epsilon \frac{(\alpha)^r}{r!} \frac{(\beta)^t}{t!}$$

$$\times \bar{H}_{P+1, Q+1}^{M, N+1} \left[ z \left| \begin{array}{l} \left( \frac{2-\omega}{2} - \rho r - \gamma t, 2\sigma; 1 \right), (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N+1, P} \\ (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M+1, Q} \left( \frac{1-\omega}{2} - 2\rho r - 2\gamma t, \sigma; 1 \right) \end{array} \right. \right] \quad (4.2)$$

Now, multiplying by  $e^{imx}$  both sides in Eq. (4.1) and integrating it with respect to  $x$  from  $0$  to  $\pi$  and finally, making an application to Eqs. (1.9), (1.10) and (2.1), one derives

$$B_p = \frac{e^{ip\pi/2}}{2^{\omega-1}} \sum_{r, t=0}^{\infty} \epsilon \frac{(\alpha/4^\rho)^r}{r!} \frac{(\beta/4^\gamma)^t}{t!}$$

$$\times \bar{H}_{P+1, Q+2}^{M, N+1} \left[ \frac{z}{4^\sigma} \left| \begin{array}{l} (1 - \omega - 2\rho r - 2\gamma t, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N+1, P} \\ (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M+1, Q} \left( \frac{1 - \omega - 2\rho r - 2\gamma t \pm m}{2}, \sigma; 1 \right) \end{array} \right. \right] \quad (4.3)$$

Using Eqs. (4.2), (4.3), from (4.1) one obtains the required Cosine Fourier Series.

$$\begin{aligned}
& (\sin x)^{w-1} F_{G;H;H'}^{E;F;F'} \left[ \begin{matrix} (e); (f); (f'); & \alpha(\sin x)^{2\rho} \\ (g); (h); (h'); & \beta(\sin x)^{2\gamma} \end{matrix} \right] \\
& \times \bar{H}_{M,N}^{P,Q} \left[ z(\sin x)^{2\sigma} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] = \frac{1}{\sqrt{(\pi)}} \sum_{r,t=0}^{\infty} \epsilon \frac{(\alpha)^r}{r!} \frac{(\beta)^t}{t!} \\
& \times \bar{H}_{P+1,Q+1}^{M,N+1} \left[ z \left| \begin{matrix} \left(\frac{2-\omega}{2} - \rho r - \gamma t, 2\sigma; 1\right), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left(\frac{1-\omega}{2} - 2\rho r - 2\gamma t, \sigma; 1\right) \end{matrix} \right. \right] \\
& + \sum_{p=-\infty}^{\infty} \sum_{r,t=0}^{\infty} \epsilon e^{ip\pi/2} \cos px \frac{(\alpha/4^\rho)^r}{r!} \frac{(\beta/4^\gamma)^t}{t!} \frac{1}{2^{\omega-2}} \\
& \times \bar{H}_{P+1,Q+2}^{M,N+1} \left[ \frac{z}{4^\sigma} \left| \begin{matrix} (1-\omega-2\rho r-2\gamma t, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,N}, (b_j, \beta_j; B_j)_{M+1,Q} \left(\frac{1-\omega-2\rho r-2\gamma t+m}{2}, \sigma; 1\right) \end{matrix} \right. \right]. \quad (4.4)
\end{aligned}$$

## SINE FOURIER SERIES

Let

$$\begin{aligned}
f(x) &= (\sin x)^{w-1} F_{G;H;H'}^{E;F;F'} \left[ \begin{matrix} (e); (f); (f'); & \alpha(\sin x)^{2\rho} \\ (g); (h); (h'); & \beta(\sin x)^{2\gamma} \end{matrix} \right] \\
& \times \bar{H}_{M,N}^{P,Q} \left[ z(\sin x)^{2\sigma} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] = \sum_{p=-\infty}^{\infty} C_p \sin px. \quad (5.1)
\end{aligned}$$

Multiplying by  $e^{imx}$  both sides in Eq. (5.1) and then integrating it with respect to  $x$  from  $0$  to  $\pi$  and making to Eqs. (1.11) and (2.1), one obtains

$$\begin{aligned}
C_p &= \frac{e^{ip\pi/2}}{2^{\omega-1}} \sum_{r,t=0}^{\infty} \epsilon \frac{(\alpha/4^\rho)^r}{r!} \frac{(\beta/4^\gamma)^t}{t!}. \\
& \times \bar{H}_{P+1,Q+2}^{M,N+1} \left[ \frac{z}{4^\sigma} \left| \begin{matrix} (1-\omega-2\rho r-2\gamma t, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left(\frac{1-\omega-2\rho r-2\gamma t+m}{2}, \sigma; 1\right) \end{matrix} \right. \right]. \quad (5.2)
\end{aligned}$$

Now making an application of Eqs. (5.1) and (5.2) one finds required Sine Fourier Series.

$$\begin{aligned}
& (2 \sin x)^{w-1} F_{G;H;H'}^{E;F;F'} \left[ \begin{matrix} (e); (f); (f'); & \alpha(\sin x)^{2\rho} \\ (g); (h); (h'); & \beta(\sin x)^{2\gamma} \end{matrix} \right] \\
& \times \bar{H}_{M,N}^{P,Q} \left[ z(\sin x)^{2\sigma} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] \\
& = \sum_{p=-\infty}^{\infty} \sum_{r,t=0}^{\infty} \frac{2 \epsilon e^{ip\pi/2}}{i} \sin px \epsilon \frac{(\alpha/4^p)^r}{r!} \frac{(\beta/4^t)^t}{t!} \\
& \times \bar{H}_{P+1,Q+2}^{M,N+1} \left[ \frac{z}{4^\sigma} \left| \begin{matrix} (1-\omega-2pr-2\gamma t, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1-\omega-2pr-2\gamma t \pm m}{2}, \sigma; 1 \right) \end{matrix} \right. \right]. \quad (5.3)
\end{aligned}$$

## MULTIPLE EXPONENTIAL FOURIER SERIES

Let

$$\begin{aligned}
f(x_1, \dots, x_n) &= (\sin x)^{w_1-1} \dots (\sin x)^{w_n-1} F_{G_1;H_1;H'_1}^{E_1;F_1;F'_1} \left[ \begin{matrix} (e_1); (f_1); (f'_1); & \alpha_1(\sin x_1)^{2\rho_1} \\ (g_1); (h_1); (h'_1); & \beta_1(\sin x_1)^{2\gamma_1} \end{matrix} \right] \\
& \dots F_{G_n;H_n;H'_n}^{E_n;F_n;F'_n} \left[ \begin{matrix} (e_n); (f_n); (f'_n); & \alpha_n(\sin x_n)^{2\rho_n} \\ (g_n); (h_n); (h'_n); & \beta_n(\sin x_n)^{2\gamma_n} \end{matrix} \right] \\
& \times \bar{H}_{M,N}^{P,Q} \left[ z(\sin x_1)^{2\sigma_1} \dots (\sin x_n)^{2\sigma_n} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] \\
& = \sum_{p_1=-\infty}^{\infty} \dots \sum_{p_n=-\infty}^{\infty} A_{p_1 \dots p_n} e^{-i(p_1 x_1 + \dots + p_n x_n)}. \quad (6.1)
\end{aligned}$$

Equation (6.1) is valid, since  $f(x_1, \dots, x_n)$  is continuous and of bounded variation in the open interval  $(0, \pi)$ . In the series (6.1), to calculate  $A_{p_1, \dots, p_n}$  we fix  $x_1, \dots, x_{n-1}$ , so that

$$\sum_{p_1=-\infty}^{\infty} \dots \sum_{p_{n-1}=-\infty}^{\infty} A_{p_1 \dots p_{n-1}} e^{-i(p_1 x_1 + \dots + p_{n-1} x_{n-1})}$$

depends only on  $p_n$ .

Furthermore, it must be the coefficient of Fourier exponential series in  $x_n$  of  $f(x_1, \dots, x_n)$  over  $0 < x_n < \pi$ .

Now multiplying by  $e^{im_n x_n}$  both sides in Eq. (6.1) and integrating with respect to  $x_n$  from 0 to  $\pi$ , one gets

$$\begin{aligned}
 & (\sin x_1)^{w_1-1} \dots (\sin x_n)^{w_n-1} F_{G_1; H_1; H'_1}^{E_1; F_1; F'_1} \left[ \begin{matrix} (e_1); (f_1); (f'_1); & \alpha_1 (\sin x_1)^{2\rho_1} \\ (g_1); (h_1); (h'_1); & \beta_1 (\sin x_1)^{2\gamma_1} \end{matrix} \right] \\
 & \dots F_{G_{n-1}; H_{n-1}; H'_{n-1}}^{E_{n-1}; F_{n-1}; F'_{n-1}} \left[ \begin{matrix} (e_{n-1}); (f_{n-1}); (f'_{n-1}); & \alpha_{n-1} (\sin x_{n-1})^{2\rho_{n-1}} \\ (g_{n-1}); (h_{n-1}); (h'_{n-1}); & \beta_{n-1} (\sin x_{n-1})^{2\gamma_{n-1}} \end{matrix} \right] \\
 & \times \int_0^\pi (\sin x_n)^{w_n-1} e^{im_n x_n} F_{G_n; H_n; H'_n}^{E_n; F_n; F'_n} \left[ \begin{matrix} (e_n); (f_n); (f'_n); & \alpha_n (\sin x_n)^{2\rho_n} \\ (g_n); (h_n); (h'_n); & \beta_n (\sin x_n)^{2\gamma_n} \end{matrix} \right] \\
 & \times \bar{H}_{M,N}^{P,Q} \left[ \begin{matrix} z (\sin x_1)^{2\sigma_1} \dots (\sin x_n)^{2\sigma_n} & (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ & (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right] dx_n \\
 & = \sum_{p_1=-\infty}^{\infty} \dots \sum_{p_{n-1}=-\infty}^{\infty} A_{p_1 \dots p_{n-1}} e^{-i(p_1 x_1 + \dots + p_n x_n)} + \sum_{p_n=-\infty}^{\infty} \int_0^\pi (e^{i(m_n - p_n)x_n} dx \quad (6.2)
 \end{aligned}$$

Using Eqs. (1.9) and (2.1), from Eq. (6.2), respectively, one finds

$$\begin{aligned}
 A_{p_1 \dots p_n} & = \sum_{r_1, t_1=0}^{\infty} \dots \sum_{r_n, t_n=0}^{\infty} \frac{e^{i(p_1 + \dots + p_n)\pi/2}}{2^{(\omega_1 + \dots + \omega_n) - n}} (\epsilon_1 \dots \epsilon_n) \\
 & \times \frac{(\alpha_1/4^{\rho_1})^{r_1}}{r_1!} \frac{(\beta_1/4^{\gamma_1})^{t_1}}{t_1!}, \dots, \frac{(\alpha_n/4^{\rho_n})^{r_n}}{r_n!} \frac{(\beta_n/4^{\gamma_n})^{t_n}}{t_n!} \\
 & \times \bar{H}_{P+1, Q+1}^{M, N+1} \left[ \begin{matrix} \frac{z}{4^{(\sigma_1 + \dots + \sigma_n)}} & (1 - \omega_1 - 2\rho_1 r_1 - 2\gamma_1 t_1, 2\sigma_1; 1) \dots (1 - \omega_n - 2\rho_n r_n - 2\gamma_n t_n, 2\sigma_n; 1), \\ & (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1 - \omega_1 - 2\rho_1 r_1 - 2\gamma_1 t_1 \pm m_1}{2}, \sigma_1; 1 \right) \\ & (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ & \dots \left( \frac{1 - \omega_n - 2\rho_n r_n - 2\gamma_n t_n \pm m_n}{2}, \sigma_n; 1 \right) \end{matrix} \right]. \quad (6.3)
 \end{aligned}$$

Using Eq. (6.3) into Eq. (6.1), required multiple exponential Fourier series is obtained.

Let

$$\begin{aligned}
 & (\sin x_1)^{w_1-1} \dots (\sin x_n)^{w_n-1} F_{G;H;H'}^{E;F;F'} \left[ \begin{matrix} (e_1); (f_1); (f'_1); & \alpha_1(\sin x_1)^{2\rho_1} \\ (g_1); (h_1); (h'_1); & \beta_1(\sin x_1)^{2\gamma_1} \end{matrix} \right] \\
 & \quad \times F_{G_n;H_n;H'_n}^{E_n;F_n;F'_n} \left[ \begin{matrix} (e_n); (f_n); (f'_n); & \alpha_n(\sin x_n)^{2\rho_n} \\ (g_n); (h_n); (h'_n); & \beta_n(\sin x_n)^{2\gamma_n} \end{matrix} \right] \\
 & \quad \times \bar{H}_{M,N}^{P,Q} \left[ \begin{matrix} z(\sin x_1)^{2\sigma_1} \dots (\sin x_n)^{2\sigma_n} & (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ & (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right] \\
 & = \sum_{p_1 \dots p_n = -\infty}^{\infty} \dots \sum_{r_1 \dots r_n, t_1 \dots t_n = 0}^{\infty} \frac{(\epsilon_1 \dots \epsilon_n)}{2^{(\omega_1 + \dots + \omega_n) - n}} \times e^{-i(p_1 n_1 + \dots + p_n n_n)} \cdot e^{(p_1 + \dots + p_n) N_2} \\
 & \quad \frac{(\alpha_1/4^{\rho_1})^{r_1}}{r_1!} \frac{(\beta_1/4^{\gamma_1})^{t_1}}{t_1!}, \dots, \frac{(\alpha_n/4^{\rho_n})^{r_n}}{r_n!} \frac{(\beta_n/4^{\gamma_n})^{t_n}}{t_n!} \\
 & \quad \times \bar{H}_{P+n, Q+n}^{M, N+n} \left[ \begin{matrix} \frac{z}{4^{(\sigma_1 + \dots + \sigma_n)}} & (1 - \omega_1 - 2\rho_1 r_1 - 2\gamma_1 t_1, 2\sigma_1; 1) \dots (1 - \omega_n - 2\rho_n r_n - 2\gamma_n t_n, 2\sigma_n; 1), \\ & (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1 - \omega_1 - 2\rho_1 r_1 - 2\gamma_1 t_1 \pm m_1}{2}, \sigma_1; 1 \right) \\ & (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ & \dots \left( \frac{1 - \omega_n - 2\rho_n r_n - 2\gamma_n t_n \pm m_n}{2}, \sigma_n; 1 \right) \end{matrix} \right]. \tag{6.4}
 \end{aligned}$$

## PARTICULAR CASES

Setting  $\beta_1, \dots, \beta_n = 0$  in Eq. (2.2), L.H.S. of Eq. (7.1) is obtained:

$$\begin{aligned}
 & \int_0^\pi \dots \int_0^\pi (\sin x_1)^{w_1-1} \dots (\sin x_n)^{w_n-1} e^{i(m_1 x_1 + \dots + m_n x_n)} \\
 & \quad \times_{E_1+F_1} F_{G_1+H_1} \left[ \begin{matrix} (e_1); (f_1); & \alpha_1(\sin x_1)^{2\rho_1} \\ (g_1); (h_1); & \end{matrix} \right] \dots \times_{E_n+F_n} F_{G_n+H_n} \left[ \begin{matrix} (e_n); (f_n); & \alpha_n(\sin x_n)^{2\rho_n} \\ (g_n); (h_n); & \end{matrix} \right] \\
 & \quad \times \bar{H}_{M,N}^{P,Q} \left[ \begin{matrix} z(\sin x_1)^{2\sigma_1} \dots (\sin x_n)^{2\sigma_n} & (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ & (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right] dx_1 \dots dx_n
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(\pi)^n e^{i(m_1+\dots+m_n)\pi/2}}{2^{(\omega_1+\dots+\omega_n)-n}} \sum_{r_1 \dots r_n=0}^{\infty} \frac{\prod_{k_1=1}^{E_1} (e_{1k_1})_{r_1} \prod_{k_1=1}^{F_1} (f_{1k_1})_{r_1}}{\prod_{k_1=1}^{G_1} (g_{1k_1})_{r_1} \prod_{k_1=1}^{H_1} (h_{1k_1})_{r_1}} \dots \\
&\dots \frac{\prod_{k_n=1}^{E_n} (e_{nk_n})_{r_n} \prod_{k_n=1}^{F_n} (f_{nk_n})_{r_n} (\alpha_1/4^{\rho_1})_{r_1} \dots (\alpha_n/4^{\rho_n})_{r_n}}{\prod_{k_n=1}^{G_n} (g_{nk_n})_{r_n} \prod_{k_n=1}^{H_n} (h_{nk_n})_{r_n} r_1! \dots r_n!} \\
&\times \overline{H}_{P+2, Q+2}^{M, N+n} \left[ \begin{array}{c} \left| \frac{z}{4^{(\sigma_1+\dots+\sigma_n)}} \right| (1-\omega_1-2\rho_1 r_1, 2\sigma_1; 1) \dots (1-\omega_n-2\rho_n r_n, 2\sigma_n; 1), \\ (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M+1, Q} \left( \frac{1-\omega_1-2\rho_1 r_1 \pm m_1}{2}, \sigma_1; 1 \right) \\ \\ (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N+1, P} \\ \dots \left( \frac{1-\omega_n-2\rho_n r_n \pm m_n}{2}, \sigma_n; 1 \right) \end{array} \right]. \quad (7.1)
\end{aligned}$$

Further setting  $\alpha_1, \dots, \alpha_n = 0$  in Eq. (7.1), one gets L.H.S. of Eq. (7.2)

$$\begin{aligned}
&\int_0^\pi \dots \int_0^\pi (\sin x_1)^{w_1-1} \dots (\sin x_n)^{w_n-1} e^{i(m_1 x_1 + \dots + m_n x_n)} \\
&\times \overline{H}_{P, Q}^{M, N} \left[ \begin{array}{c} \left| z (\sin x_1)^{2\sigma_1} \dots (\sin x_n)^{2\sigma_n} \right| (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N+1, P} \\ (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M+1, Q} \end{array} \right] dx_1 \dots dx_n \\
&= \frac{(\pi)^n e^{i(m_1+\dots+m_n)\pi/2}}{2^{(\omega_1+\dots+\omega_2)-n}} \\
&\times \overline{H}_{P+n, Q+n}^{M, N+1} \left[ \begin{array}{c} \left| \frac{z}{4^{(\sigma_1+\dots+\sigma_n)}} \right| (1-\omega_1, 2\sigma_1; 1) \dots (1-\omega_n, 2\sigma_n; 1), \\ (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M+1, Q} \left( \frac{1-\omega_1 \pm m_1}{2}, \sigma_1; 1 \right) \\ \\ (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N+1, P} \\ \dots \left( \frac{1-\omega_n \pm m_n}{2}, \sigma_n; 1 \right) \end{array} \right]. \quad (7.2)
\end{aligned}$$

Now setting  $\alpha = \beta = 0$  in Eq. (3.3) one establishes

$$\begin{aligned}
& (\sin x)^{w-1} \bar{H}_{P,Q}^{M,N} \left[ z(\sin x)^{2\sigma} \left| \begin{array}{l} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{array} \right. \right] \\
&= \sum_{p=-\infty}^{\infty} \frac{e^{ip(\frac{\pi}{2}-x)}}{2^{w-1}} \bar{H}_{P+1,Q+2}^{M,N+1} \left[ \frac{z}{4^\sigma} \left| \begin{array}{l} (1-\omega, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1-\omega \pm p}{2}, \sigma; 1 \right) \end{array} \right. \right]. \quad (7.3)
\end{aligned}$$

Letting  $p = 2l$  as  $l$  is an integer, from Eq. (7.3), it can be established that

$$\begin{aligned}
L.H.S. \text{ of (7.3)} &= \frac{1}{\sqrt{\pi}} \bar{H}_{P+1,Q+1}^{M,N+1} \left[ z \left| \begin{array}{l} \left( \frac{2-\omega}{2}, \sigma; 1 \right), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1-\omega}{2}, \sigma; 1 \right) \end{array} \right. \right] \\
&+ \frac{1}{2^{w-2}} \sum_{p_n=1}^{\infty} \cos l\pi \cos 2lx \bar{H}_{P+1,Q+2}^{M,N+1} \left[ \frac{z}{4^\sigma} \left| \begin{array}{l} (1-\omega, \sigma; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1-\omega \pm 2l}{2}, \sigma; 1 \right) \end{array} \right. \right] \quad (7.4)
\end{aligned}$$

Further letting  $p = (2l + 1)$  as  $l$  is an integer, from Eq. (7.3) one obtains L.H.S. of Eq. (7.4)

$$\begin{aligned}
&= \frac{1}{2^{w-2}} \sum_{p=1}^{\infty} \sin(2l+1)\pi/2 \cdot \sin(2l+1)x \\
&\times \bar{H}_{P+1,Q+2}^{M,N+1} \left[ \frac{z}{4^\sigma} \left| \begin{array}{l} (1-\omega, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \left( \frac{1-\omega \pm (2l+1)}{2}, \sigma; 1 \right) \end{array} \right. \right] \quad (7.5)
\end{aligned}$$

Similarly, remaining particular cases can be evaluated by Eqs. (4.4) and (5.3) by applying the same techniques.

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