

# THE GENERALIZATION OF TWO VARIABLES HERMITE POLYNOMIALS

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## Abstract

The object of this paper is to obtain some generalization of two variables Hermite polynomials, generating functions, recurrence relations, partial differential equations, Rodrigues formula, relationships with Hermite and Legendre polynomials of one variable, and some special properties.

**Keywords:** Generating function, Hermite polynomial, Legendre polynomial.

## INTRODUCTION

Generalized polynomial set  $H_{n,m,v}(x)$  has been defined by Lahiri [1971] in the form

$$\sum_{n=0}^{\infty} \frac{H_{n,m,v}(x) t^n}{n!} = e^{vxt - t^m}$$

where  $n$  is a nonnegative and  $m$  is positive integer. Recently, Khan and Abukahmmash [1998] defined Hermite polynomials of two variables  $H_n(x, y)$  as follows:

$$H_n(x, y) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n! (-y)^r H_{n-2r}(x)}{r! (n-2r)!}$$

where  $H_n(x)$  are the well known Hermite polynomials of one variable.

They defined generating functions for  $H_n(x, y)$  in the following form:

$$\sum_{n=0}^{\infty} \frac{H_n(x, y) t^n}{n!} = e^{2xt - (y+1)t^2}$$

For brevity, following notations are being listed:

$$\Delta_k [b; a] = \left( \frac{a}{b} \right)_k, \left( \frac{a+1}{b} \right)_k, \dots, \left( \frac{a+b-1}{b} \right)_k$$

$$\Delta(b; a) = \frac{a}{b}, \frac{a+1}{b}, \dots, \frac{a+b-1}{b}$$

In this paper a generalization of the Hermite polynomials of two variables and many results have been obtained.

### GENERATING FUNCTION FOR $H_{n,m,\nu}(x, y)$

Generalized Hermite polynomials of two variables have been defined by means of generating relation

$$\sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x, y) t^n}{n!} = e^{\nu x t - (y+1)t^m} \quad (1)$$

where  $n$  is a nonnegative integer and  $m$  is a positive integer. Now, consider

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x, y) t^n}{n!} &= \sum_{n,k=0}^{\infty} \frac{(-1)^k (\nu x)^n t^{n+mk} (y+1)^k}{n! k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(\nu x)^{n-mk} (-1)^k (y+1)^k t^n}{(n-mk)! k!} \end{aligned}$$

On comparing coefficients of  $t^n$  from both the sides, we get

$$\begin{aligned} H_{n,m,\nu}(x, y) &= \sum_{k=0}^{[n/m]} \frac{(-1)^k n! (\nu x)^{n-mk} (y+1)^k}{(n-mk)! k!} \\ &= \sum_{k=0}^{[n/m]} \frac{(-1)^k (\nu x)^{n-mk} (y+1)^k (-n)_{mk} n!}{k! (-1)^{mk} n!} \\ &= \sum_{k=0}^{[n/m]} \frac{(-1)^{(m+1)k} m^{mk} \Delta_k [(m; n)] (\nu x)^{n-mk} (y+1)^k}{k!} \quad (2) \end{aligned}$$

$$= (\nu x)^n {}_m F_0 \left[ \Delta(m; n); -; -(y+1) \left( \frac{-m}{\nu x} \right)^m \right] \quad (3)$$

Particular case

1) Setting  $y=0$  in (2) we get a known generating relation ([Lahiri 1971]; eq. (3.2)),

$$\begin{aligned}
 H_{n,m,v}(x, o) &= (vx)^n {}_mF_0 \left[ \Delta(m; n); -; - \left( \frac{-m}{vx} \right)^m \right] \\
 &= H_{n,m,v}(x)
 \end{aligned}$$

2) Setting  $m=1$  , we get

$$H_{n,1,v}(x, y) = (vx - (y + 1))^n$$

Next, consider the sum

$$\begin{aligned}
 &\sum_{n=0}^{\infty} \frac{(c)_n H_{n,m,v}(x, y) t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (c)_n n! (vx)^{n-mk} (y + 1)^k t^n}{(n - mk)! k!} \\
 &= \sum_{n,k=0}^{\infty} \frac{(-1)^k (c)_{mk} (c + mk)_n (vx)^n (y + 1)^k t^{n+mk}}{n! k!} \\
 &= \sum_{k=0}^{\infty} {}_1F_0 [c + mk; -; (vxt)] \frac{m^{mk} \Delta_k(m; c) (-(y + 1)t^m)^k}{k!}
 \end{aligned}$$

Thus one arrives at the divergent generating relation.

$$\begin{aligned}
 &(1 - vxt)^{-c} {}_mF_0 \left[ \Delta(m; c); -; - (y + 1) \left( \frac{mt}{1 - vxt} \right)^m \right] \\
 &\cong \sum_{n=0}^{\infty} \frac{(c)_n H_{n,m,v}(x, y) t^n}{n!} \tag{4}
 \end{aligned}$$

Particular case, when  $y=0$  ,  $m=v=2$  , a well-known result is obtained

$$\begin{aligned}
 &(1 - 2xt)^{-c} {}_2F_0 \left[ \Delta(2; c); -; - (y + 1) \left( \frac{2t}{1 - 2xt} \right)^2 \right] \\
 &\cong \sum_{n=0}^{\infty} \frac{(c)_n H_n(x, y) t^n}{n!}
 \end{aligned}$$

Next consider

$$\begin{aligned}
 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_{n+k,m,v}(x, y) t^n v^k}{k! n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{H_{n,m,v}(x, y) t^{n-k} v^k}{k! (n - k)!} \\
 &= \sum_{k=0}^{\infty} \frac{H_{n+k,m,v}(x, y)}{n!} \sum_{k=0}^n \frac{n! t^{n-k} v^k}{k! (n - k)!}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x,y)(t+v)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x,y)}{n!} \sum_{k=0}^n \frac{n! t^{n-k} v^k}{k!(n-k)!} \\
&= e^{vx(t+v)-(y+1)(t+v)^m}
\end{aligned}$$

for  $m=v=2$ , one gets

$$= e^{2xt-(y+1)t^2} \sum_{k=0}^{\infty} \frac{H_{k,m,v}(x-yt-t,y)v^k}{k!}$$

By equating the coefficients of  $\frac{v^k}{k!}$ , one obtains

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(x,y)t^n}{n!} = e^{2xt-(y+1)t^2} H_{k,m,v}(x-yt-t,y) \quad (5)$$

Consider,

$$\begin{aligned}
&\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} \frac{H_{n+r+s,m,v}(x,y)t^n u^s v^r}{r!s!n!} \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{H_{n+r+s,m,v}(x,y)t^n u^s v^r}{r!s!n!} \\
&= \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x,y)t^n}{n!} \sum_{r=0}^n \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (-1)^{r+s} \left(\frac{u}{t}\right)^s \left(\frac{v}{t}\right)^r}{r!s!} \\
&= \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x,y)t^n (t+u+v)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x,y)t^n}{n!} \left(1 + \frac{u}{t} + \frac{v}{t}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x,y)t^n (t+u+v)^n}{n!} \quad (6) \\
&= e^{vx(t+u+v)-(y+1)(t+u+v)^m}
\end{aligned}$$

Consider the identity,

$$\begin{aligned}
e^{vx t-(y+1)t^m} &= e^{v(xt)-(y+1)(xt)^m} e^{(y+1)x^m t^m-(y+1)t^m} \\
&= e^{v(xt)-(y+1)(xt)^m} e^{(y+1)(x^m-1)t^m}
\end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x,y)}{n!} &= \sum_{n=0}^{\infty} \frac{H_{n,m,1}(1,y)(xt)^n}{n!} \cdot \sum_{k=0}^{\infty} \frac{(y+1)^k (x^2-1)^k t^{mk}}{k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{H_{n,m,1}(1,y)(xt)^n (y+1)^k (x^2-1)^k t^{mk}}{n!k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n/m} \frac{H_{n,m,1}(1,y)(xt)^{n-mk} (y+1)^k (x^2-1)^k t^{mk}}{(n-mk)!k!} \end{aligned}$$

Equating the coefficient of  $t^n$  one gets

$$H_{n,m,\nu}(x,y) = \sum_{k=0}^{n/m} \frac{n! H_{n,m,\nu}(1,y)(xt)^{n-mk} t^{n-xk} (y+1)^k (x^2-1)^k}{(n-mk)!k!} \quad (7)$$

Next, consider the identity

$$e^{\nu(x_1+x_2)t^m} = e^{\nu x_1 t - (y+1)t^m} e^{\nu x_2 t - (y+1)t^m} e^{(y+1)t^m}$$

Or

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x_1+x_2,y)t^n}{n!} &= \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x_1,y)t^n}{n!} \sum_{r=0}^{\infty} \frac{H_{r,m,\nu}(x_2,y)t^r}{r!} \sum_{s=0}^{\infty} \frac{(y+1)^s t^{ms}}{s!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{H_{n,m,\nu}(x_1,y)H_{r,m,\nu}(x_2,y)(y+1)^s t^{n+r+ms}}{n!r!s!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\lfloor n/m \rfloor} \frac{H_{n-ms,m,\nu}(x_1,y)H_{r,m,\nu}(x_2,y)(y+1)^s t^{n+r}}{(n-ms)!r!s!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^n \sum_{s=0}^{\lfloor \frac{n-r}{m} \rfloor} \frac{H_{n-r-ms,m,\nu}(x_1,y)H_{r,m,\nu}(x_2,y)(y+1)^s t^{n+r}}{(n-r-ms)!r!s!} \end{aligned}$$

Equating the coefficient of  $t^n$  one obtains

$$\begin{aligned} H_{n,m,\nu}(x_1+x_2,y) &= \sum_{r=0}^n \sum_{s=0}^{\lfloor \frac{n-r}{m} \rfloor} \frac{n! H_{n-r-ms,m,\nu}(x_1,y)H_{r,m,\nu}(x_2,y)(y+1)^s}{(n-r-ms)!r!s!} \quad (8) \end{aligned}$$

Next, consider the identity

$$e^{\nu x t - (y_1 + y_2 + 1)t^m} \cdot e^{\nu x t - t^m} = e^{\nu(xt) - (y_1 + 1)t^m} e^{\nu x t - (y_2 + 1)t^m}$$

$$\sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x, y_1 + y_2) t^m}{n!} \sum_{k=0}^{\infty} \frac{H_{k,m,\nu}(x) t^k}{k!} = \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x, y_1) t^n}{n!} \cdot \sum_{k=0}^{\infty} \frac{H_{k,m,\nu}(x, y_2) t^k}{k!}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} H_{n,m,\nu}(x, y_1 + y_2) H_{n,m,\nu}(x) t^{n+k} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{H_{n,m,\nu}(x, y_1) H_{k,m,\nu}(x, y_2) t^{n+k}}{n! k!}$$

Equating the coefficient of  $t^{n+k}$  one gets,

$$H_{n,m,\nu}(x, y_1 + y_2) H_{n,m,\nu}(x) = H_{n,m,\nu}(x, y_1) H_{k,m,\nu}(x, y_2) \quad (9)$$

where  $H_{n,m,\nu}(x)$  is known as generalized Hermite polynomial ([Lahiri 1971], Eq. (3.1)).

Now, by considering the identity

$$e^{\nu \lambda x t - (y+1)t^m} = e^{\nu x t - (y+1)t^m} e^{\nu(\lambda-1)xt}$$

$$H_{n,m,\nu}(\lambda x, y) = \sum_{k=0}^n \frac{n! H_{n-k,m,\nu}(\lambda x, y) m^\lambda (\lambda-1)^k (x)^k}{k!(n-k)!} \quad (10)$$

Similarly, by considering the identity

$$e^{\nu x t - (\lambda y + 1)t^m} = e^{\nu x t - (y+1)t^m} e^{(1-\lambda)y t^m}$$

one obtains

$$H_{n,m,\nu}(x, \lambda y) = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{n! H_{n-mk,m,\nu}(x, y) (1-y)^k y^k}{k!(n-mk)!} \quad (11)$$

In a similar way, by considering the identity

$$e^{\nu \lambda x t - (\mu y + 1)t^m} = e^{\nu x t - (y+1)t^m} e^{\nu(\lambda-1)xt - (1-\mu)t^m} e^{t^m}$$

one obtains

$$H_{n,m,\nu}(\lambda x, \mu y)$$

$$= \sum_{r=0}^n \sum_{s=0}^{\lfloor \frac{n-r}{m} \rfloor} \frac{n! H_{n-r-ms,m,\nu}(x, y) H_{r,m,\nu}\{(\lambda-1)x, (1-\mu)y\}}{r! s! (n-r-ms)!} \quad (12)$$

## RECURRENCE RELATIONS

Since  $e^{\nu x t - (y+1)t^m} = \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x, y) t^n}{n!}$  (13)

by differentiating Eq. (13) partially w.r.t.  $x$ , it becomes

$$\nu t e^{\nu x t - (y+1)t^m} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial x} H_{n,m,\nu}(x, y) \quad (14)$$

Now Eq. (14) can be written as

$$\sum_{n=0}^{\infty} \frac{\nu H_{n,m,\nu}(x,y)t^{m+1}}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial x} H_{n,m,\nu}(x,y)$$

which with a shift of index on the left, yields  $\frac{\partial}{\partial x} H_{0,m,\nu}(x,y)=0$  and for  $n \geq 1$ ,

$$\frac{\nu H_{n-1,m,\nu}(x,y)}{(n-1)!} = \frac{1}{n!} \frac{\partial}{\partial x} H_{n,m,\nu}(x,y) \tag{15}$$

or

$$\frac{\partial}{\partial x} H_{n,m,\nu}(x,y) = \nu n H_{n-1,m,\nu}(x,y) \tag{16}$$

Iteration of Eq. (16) gives

$$\frac{\partial^s}{\partial x^s} H_{n,m,\nu}(x,y) = \frac{\nu^s n! H_{n-s,m,\nu}(x,y)}{(n-s)!} \tag{17}$$

Differentiating Eq. (13) partially w.r.t.  $y$ , one gets

$$-t^m e^{\nu xt - (y+1)t^m} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial y} H_{n,m,\nu}(x,y)$$

or

$$\sum_{n=0}^{\infty} \frac{-H_{n,m,\nu}(x,y)t^{n+m}}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial y} H_{n,m,\nu}(x,y) \tag{18}$$

This with a shift of index on the left yields:

$$\frac{\partial}{\partial y} H_{0,m,\nu}(x,y) = 0, \quad \frac{\partial}{\partial x} H_{1,m,\nu}(x,y) = 0, \text{ for } n \geq 2,$$

$$\frac{-H_{n-2,m,\nu}(x,y)}{(n-2)!} = \frac{1}{n!} \frac{\partial}{\partial y} H_{n,m,\nu}(x,y)$$

$$\frac{\partial^r}{\partial y^r} H_{n,m,\nu}(x,y) = -n(n-1)H_{n-2,m,\nu}(x,y) \tag{19}$$

Iteration of Eq. (19) gives

$$\frac{\partial^r}{\partial y^r} H_{n,m,\nu}(x,y) = \frac{(-1)^r}{(n-2r)!} H_{n-2r,m,\nu}(x,y) \tag{20}$$

Differentiating Eq. (13) partially w. r. t.  $t$  gives

$$\nu x - m(y+1)t^{m-1} e^{\nu xt - (y+1)t^m} = \sum_{n=0}^{\infty} \frac{n H_{n,m,\nu}(x,y)t^{n-1}}{n!}$$

$$\left( vx - my + m t^{m-1} \right) e^{vxt - (y+1)t^m} = \sum_{n=1}^{\infty} \frac{n H_{n,m,v}(x, y) t^{n-1}}{(n-1)!} \quad (21)$$

Multiplying Eq. (14), (18) and (21) by  $x$ ,  $m(y+1)$  and  $-t$ , respectively and adding them gives

$$x \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial x} H_{n,m,v}(x, y) + m(y+1) \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial y} H_{n,m,v}(x, y) - \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_{n,m,v}(x, y) = 0$$

or

$$\sum_{n=0}^{\infty} \frac{x t^n}{n!} \frac{\partial}{\partial x} H_{n,m,v}(x, y) + m \sum_{n=0}^{\infty} \frac{(y+1) t^n}{n!} \frac{\partial}{\partial y} H_{n,m,v}(x, y) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_{n,m,v}(x, y)$$

Equating the coefficients of  $t^n$ , one gets

$$\frac{x}{n!} \frac{\partial}{\partial x} H_{n,m,v}(x, y) + m \frac{(y+1)}{n!} \frac{\partial}{\partial y} H_{n,m,v}(x, y) = \frac{1}{(n-1)!} H_{n,m,v}(x, y)$$

or

$$x \frac{\partial}{\partial x} H_{n,m,v}(x, y) + m(y+1) \frac{\partial}{\partial y} H_{n,m,v}(x, y) = n H_{n,m,v}(x, y) \quad (22)$$

Combination of Eqs. (4), (7) and (10) yields pure recurrence relation:

$$2nx H_{n-1,m,v}(x, y) - nm(y+1)(n-1)(n-2)\dots(n-(m-1)) H_{n-1,m,v}(x, y) = n H_{n,m,v}(x, y) \quad (23)$$

and from Eq. (16) and (23) partial differential equation can be obtained:

$$2nx \left( \frac{1}{2n} \frac{\partial}{\partial x} H_{n,m,v}(x, y) \right) - nm(y+1)(n-1)(n-2)\dots(n-(m-1)) H_{n-1,m,v}(x, y) = n H_{n,m,v}(x, y)$$

$$x \frac{\partial}{\partial x} H_{n,m,v}(x, y) - nm(y+1)(n-1)(n-2)\dots(n-(m-1)) H_{n-1,m,v}(x, y) = n H_{n,m,v}(x, y) \quad (24)$$

## RELATIONSHIP BETWEEN $H_{n,m,v}(x)$ AND $H_{n,m,v}(x, y)$

Since

$$\sum_{n=0}^{\infty} \frac{H_{n,m,v}(x) t^n}{n!} = e^{vxt - t^m} \quad (25)$$

where  $H_{n,m,v}(x)$  is generalized Hermite polynomial of one variable

[Lahiri 1971]. Replacing  $x$  by  $\frac{x}{\sqrt{y+1}}$  and  $t$  by  $\sqrt{y+1} t$  in (25), one gets

$$\sum_{n=0}^{\infty} \frac{H_{n,m,v} \left( \frac{x}{\sqrt{y+1}} \right) (\sqrt{y+1})^n t^n}{n!} = e^{vxt - (y+1)t^m} \tag{26}$$

In view of Eq. (1), Eq. (26) can be rewritten as:

$$\sum_{n=0}^{\infty} H_{n,m,v}(x, y) = \sum_{n=0}^{\infty} \frac{H_{n,m,v} \left( \frac{x}{\sqrt{y+1}} \right) (y+1)^{\frac{n}{m}} t^n}{n!}$$

Equating the coefficients of  $t^n$  gives

$$H_{n,m,v}(x, y) = (y+1)^{\frac{n}{m}} H_{n,m,v} \left( \frac{x}{\sqrt{y+1}} \right) \tag{27}$$

Now

$$H_{n,m,v}(-x, y) = (-1)^n H_{n,m,v}(x, y) \tag{28}$$

For  $y=0$ , Eq. (27) reduces to

$$H_{n,m,v}(x, 0) = H_{n,m,v}(x) \tag{29}$$

and for  $x=0$ , it becomes

$$H_{n,m,v}(0, y) = (y+1)^{\frac{n}{m}} H_{n,m,v}(0) \tag{30}$$

but  $H_{mn,m,v}(0) = (-1)^n m^{mn} \left( \frac{1}{m} \right)_n$ ,  $H_{mn+1,m,v}(0) = 0$  \tag{31}

Now using Eq. (31) into (30) one obtains

$$\left. \begin{aligned} H_{mn,m,v}(0, y) &= (y+1)^n (-1)^n m^{mn} \left( \frac{1}{m} \right)_n \\ H_{mn+1,m,v}(0, y) &= 0 \end{aligned} \right\} \tag{32}$$

For  $x=y=0$ , Eq. (27) becomes

$$H_{n,m,v}(0, 0) = H_{n,m,v}(0) \tag{33}$$

which in the light of Eq. (31), gives

$$H_{mn+1,m,v}(0, 0) = (-1)^n m^{mn} \left( \frac{1}{m} \right)_n = H_{mn,m,v}(0) \tag{34}$$

$$H_{mn+1,m,v}(0, 0) = 0 = H_{mn,m,v}(0) \tag{35}$$

### THE RODRIGUES FORMULA

Examination of the defining relation

$$e^{vxt - (y+1)t^m} = \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x, y) t^n}{n!} \tag{36}$$

The application of Maclaurin's theorem gives at once

$$H_{n,m,\nu}(x, y) = \left[ \frac{d^n}{dt^n} e^{\nu xt - (y+1)t^m} \right]_{t=0}$$

The function  $e^{-\frac{x^\nu}{y+1}}$  is independent of  $t$ , so one may write as

$$e^{-\frac{x^\nu}{y+1}} H_{n,m,\nu}(x, y) = \left[ \frac{d^n}{dt^n} e^{-\left(\left(\frac{x}{\sqrt[m]{y+1}}\right) - (\sqrt[m]{y+1} t)\right)^m} \right]_{t=0}$$

Now substituting  $\left(\frac{x}{\sqrt[m]{y+1}}\right) - (\sqrt[m]{y+1} t) = \omega$  gives

$$\begin{aligned} e^{-\frac{x^\nu}{y+1}} H_{n,m,\nu}(x, y) &= (-1)^n (y+1)^{\frac{n}{m}} \left[ \frac{d^n}{d\omega^n} e^{-\omega^m} \right]_{\omega = \frac{x}{\sqrt[m]{y+1}}} \\ &= (-1)^n (y+1)^n \frac{d^n}{dx^n} e^{-\frac{x^\nu}{y+1}} \end{aligned}$$

or

$$H_{n,m,\nu}(x, y) = (-1)^n (y+1)^n e^{\frac{x^\nu}{y+1}} D^n e^{-\left(\frac{x^\nu}{y+1}\right)}, \quad D \equiv \frac{d}{dx} \quad (37)$$

i.e., a formula of the same nature as Rodrigues formula for Hermite polynomial of two variables [Khan and Abukahmmash 1998].

## EXPANSION OF POLYNOMIALS

Since

$$e^{\nu xt} = e^{(y+1)t^m} \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x, y) t^n}{n!}$$

It follows that

$$e^{\nu xt - (y+1)t^m} = \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x, y) t^n}{n!}$$

$$\begin{aligned} \text{Or } \sum_{n=0}^{\infty} \frac{(\nu xt)^n}{n!} &= \left( \sum_{n=0}^{\infty} \frac{(y+1)^n t^{mn}}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{H_{n,m,\nu}(x, y) t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{H_{n,m,\nu}(x, y) (y+1)^k t^{n+mk}}{n! k!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{H_{n-mk,m,\nu}(x,y)(y+1)^k t^n}{(n-mk)! k!}$$

Equating the coefficients of  $t^n$ , one gets

$$x^n = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{n! H_{n-mk,m,\nu}(x,y)(y+1)^k}{\nu^n (n-mk)! k!} \tag{38}$$

Employ Eq. (38) to expand the Legendre polynomial in a series of Hermite polynomials, e.g., consider the series

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(x) t^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (1/2)_{n-k} (2x)^{n-2k} t^n}{k! (n-2k)!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (1/2)_{n+k} (2x)^n t^{n+2k}}{k! n!} \end{aligned}$$

from Eq. (38) one has

$$\frac{(2x)^n}{n!} = \sum_{s=0}^{\lfloor \frac{n}{m} \rfloor} \frac{H_{n-ms,m,\nu}(x,y)(y+1)^s 2^n}{\nu^n (n-ms)! s!} \tag{39}$$

Hence one may write as

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(x) t^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(-1)^k (1/2)_{n+k} H_{n-ms,m,\nu}(x,y)(y+1)^s t^{n+2k} 2^n}{k! s! (n-ms)! \nu^n} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^k (1/2)_{n+k+ms} H_{n,m,\nu}(x,y)(y+1)^s t^{n+2k+ms} 2^{n+ms}}{k! s! n! (\nu)^{n+ms}} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^k \frac{(-1)^{k-s} (1/2)_{n+k+(m-1)s} H_{n,m,\nu}(x,y)(y+1)^s t^{n+2k+(m-2)s} 2^{n+ms}}{(k-s)! s! n! (\nu)^{n+ms}} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^k \frac{(-1)^k \left(\frac{1}{2} + n + k\right)_{(m-1)s} \left(\frac{1}{2}\right)_{n+k} (-1)^{-s} H_{n,m,\nu}(x,y)(y+1)^s t^{n+2k+(m-2)s} 2^{n+ms}}{k! (k-s)! s! n! \nu^{n+ms}} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} H_{n,m,\nu}(x,y) t^{n+2k}}{k! n! \nu^n} \sum_{s=0}^k \frac{k! (-1)^s \left(\frac{1}{2} + n + k\right)_{(m-1)s} (y+1)^s t^{(m-2)s}}{s! (k-s)! \nu^{ms}} \end{aligned}$$

Particular case, for  $\nu = m = 2$ , it can be obtained

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_{n+k} H_{n,2,2}(x,y) t^{n+2k}}{k! n!} \sum_{s=0}^k \frac{(-k)_s \left(\frac{1}{2} + n + k\right)_s (y+1)^s}{s! 2^{2s}} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{{}_2F_0\left[-k, \frac{1}{2} + n + k; \frac{(y+1)}{4}\right] (-1)^k \left(\frac{1}{2}\right)_{n+k} H_n(x,y) (y+1)^s t^{n+2k}}{k! n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{{}_2F_0\left[-k, \frac{1}{2} + n + k; -; \frac{(y+1)}{4}\right] (-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x,y) t^n}{k!(n-2k)!}
\end{aligned}$$

Equating the coefficients of  $t^n$ , one has

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{{}_2F_0\left[-k, \frac{1}{2} + n + k; -; \frac{(y+1)}{4}\right] (-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x,y) t^n}{k!(n-2k)!} \quad (40)$$

Khan and Abukahmmash [1998] defined the Legendre polynomial of two variables in the form

$$P_n(x, y) = \sum_{r=2}^{\lfloor \frac{n}{2} \rfloor} \frac{(-y)^r}{r!} P_{n-2r}(x) \quad (41)$$

Now employ Eq. (38) to expand the Legendre polynomial of two variables in series of the final generalization of two variables; Hermite polynomials of two variables. From Eq. (38) consider the series

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n(x, y) t^n &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-y)^r}{r!} P_{n-2r}(x) t^n \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-y)^r}{r!} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^s \left(\frac{1}{2}\right)_{n-s} (2x)^{n-2s} t^{n+2r}}{s!(n-2s)!} \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-y)^r (-1)^s \left(\frac{1}{2}\right)_{n+s} (2x)^n t^{n+2r+2s}}{r! s! n!} \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-y)^r (-1)^s \left(\frac{1}{2}\right)_{n+s} t^{n+2r+2s}}{r! s!} \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{H_{n-mk, m, y}(x, y) (y+1)^k 2^n}{v^n (n-ms)! k!}
\end{aligned}$$

$$\begin{aligned}
 &= \sum_n^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-y)^r (-1)^s \left(\frac{1}{2}\right)_{n+mk+s} t^{n+2r+2s+mk} H_{n,m,\nu}(x,y)(y+1)^k 2^{n+mk}}{r!s!n!k! \nu^{n+mk}} \\
 &= \sum_n^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=0}^s \frac{(-y)^r (-1)^{s-k} \left(\frac{1}{2}\right)_{n+(m-1)k+s} t^{n+2r+(m-2)k+2s} H_{n,m,\nu}(x,y)(y+1)^k 2^{n+mk}}{r!(s-k)!n!k! \nu^{n+mk}} \\
 &= \sum_n^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=0}^s \frac{(-y)^r (-1)^{s-k} \left(\frac{1}{2}+n+s\right)_{(m-1)k} \left(\frac{1}{2}\right)_{n+s} t^{n+2r+(m-2)k+2s} H_{n,m,\nu}(x,y)(y+1)^k 2^{n+mk}}{r!(s-k)!n!k! \nu^{n+mk}} \\
 &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-y)^r (-1)^s \left(\frac{1}{2}\right)_{n+s} 2^n H_{n,m,\nu}(x,y) t^{n+2r+2s}}{r!s!n! \nu^n} \\
 &\quad \cdot \sum_{k=0}^s \frac{\left(\frac{1}{2}+n+s\right)_{(m-1)k} (y+1)^k 2^{mk} t^{(m-2)k}}{(-1)^k k! \nu^{mk}}
 \end{aligned}$$

As a particular case, if  $m = \nu = 2$ , one gets

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-y)^r (-1)^s \left(\frac{1}{2}\right)_{n+s} 2^n H_{n,2,2}(x,y) t^{n+2r+2s}}{r!s!n!2^n} \sum_{k=0}^s \frac{\left(\frac{1}{2}+n+s\right)_k (y+1)^k 2^{2k}}{(-1)^k k! 2^{2k}} \\
 &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-y)^r (-1)^s \left(\frac{1}{2}\right)_{n+s} H_{n,2,2}(x,y) t^{n+2r+2s}}{r!s!n!} {}_2F_0 \left[ \begin{matrix} -S, \frac{1}{2}+n+s; \\ -; \end{matrix} -\left(\frac{y+1}{4}\right) \right] \\
 &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-y)^r (-1)^s \left(\frac{1}{2}\right)_{n-s} H_{n-2s,2,2}(x,y) t^{n+2r}}{r!s!(n-2s)!} {}_2F_0 \left[ \begin{matrix} -S, \frac{1}{2}+n-s; \\ -; \end{matrix} -\left(\frac{y+1}{4}\right) \right] \\
 &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor - r} \frac{(-y)^r (-1)^s \left(\frac{1}{2}\right)_{n-2r-s} H_{n-2r-2s,2,2}(x,y) t^n}{r!s!(n-2r-2s)!} {}_2F_0 \left[ \begin{matrix} -S, \frac{1}{2}+n-2r-s; \\ -; \end{matrix} -\left(\frac{y+1}{4}\right) \right]
 \end{aligned}$$

The final result is

$$P_n(x, y) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{n-r}{2} \rfloor} \frac{(-y)^r (-1)^s \left(\frac{1}{2}\right)_{n-2r-s} H_{n-2r-2s, 2, 2}(x, y)}{r! s! (n-2r-2s)!} {}_2F_0 \left[ \begin{matrix} -S, \frac{1}{2} + n - 2r - s; \\ -; \end{matrix} \left( \frac{y+1}{-4} \right) \right] \quad (42)$$

For  $y = 0$ , Eq. (42) reduces to the known result

$$P_n(x, y) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{{}_2F_0 \left[ \begin{matrix} -S, \frac{1}{2} + n - 2r - s; \\ -; -1 \end{matrix} \right] (-1)^s \left(\frac{1}{2}\right)_{n-s} H_{n-2s, 2, 2}(x, y)}{s! (n-2s)!} \quad (43)$$

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