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THE NUMERICAL INVERSION OF THE LAPLACE TRANSFORM IN REPRODUCING KERNEL HILBERT SPACES

M. Iqbal
Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
email: facl 126 @ saupmoo bitnet

Abstract: In this paper we have converted the Laplace transform into an integral equation of the first kind of convolution type, which is an ill-posed problem and used a statistical regularization method to solve it. The method is applied to three examples taken from Rodriguez and Seatzu [1993]. It gives a good approximation to the true solution and compares well with the method given by Rodriguez and Seatzu [1993]. The results are shown in the Table and the diagrams.

Keywords: Ill-posed problem, inversion of Laplace transform, convolution equation, cross-validation, filter function, regularization parameter.

INTRODUCTION

The Laplace transform inversion is a severely ill-posed problem in the terminology of improperly posed problems. Unfortunately, many problems of physical interest lead to Laplace transforms whose inverses are not readily expressed in terms of tabulated functions, although there exist extensive tables of transforms and their inverses. It is highly desirable, therefore, to have methods for approximate numerical inversion. However, no single method gives optimum results for all purposes and all occasions. For a detailed bibliography, the reader should consult Piessens and Branders [1971], Piessens [1975, 1976], and a review and comparison is given by Davies and Martin [1979] and Talbot [1979].

The problem of the recovery of a real function f(t) for $t \geq 0$, given its Laplace transform

$$\int_o^\infty e^{-st} f(t)dt = g(s) \tag{1.1}$$

for real values of s, is an ill-posed problem in the sense of Hadamard and is therefore affected by numerical instability. The ill-posedness of Laplace transform inversion in the case where $f \in L^2(R_+)$ and g(s) is known for all real and positive values of s, can be investigated by means of the Mellin transform [McWhirter and Pike 1978]. In practice, however g(s) is known only in a finite set of points. The case of an infinite set of equidistant points was investigated by Papoulis [1956].

Regularization methods have been discussed by Varah [1983], Brianzi and Frontini [1991], Brianzi [1994], Essah and Delves [1998]. Several other methods have been developed [Piessens and Branders 1971, Wahba 1977, Varah 1983, Ang *et al.* 1989, Brianzi and Frontini 1991, Brianzi 1994,] which, in general, work rather well even if they require a large computational cost and high precision numerics.

DESCRIPTION OF THE PROPOSED METHOD

In (1.1) for a given g(s) for $s \geq 0$ we wish to find f(t), satisfying f(t) = 0 for t < 0, so that (1.1) holds. Frequently g(s) is measured at certain points. We assume g(s) is given analytically with known f(t) so that we can measure the error in the numerical solution.

We shall convert the Laplace transform into the first kind integral equation of convolution type, with the following substitution in equation (1.1)

$$s = a^x \text{ and } t = a^y \text{ where } a > 1 \qquad (2.1)$$

Then

$$g(a^x) = \int_{-\infty}^{\infty} \log a \, e^{-a^{x-y}} f(a^{-y})\, a^{-y}\, dy \qquad (2.2)$$

Multiplying both sides of (2.2) by a^x we obtain the convolution equation,

$$\int_{-\infty}^{\infty} K(x-y)\, F(y)\, dy = G(x), \qquad -\infty \leq x \leq \infty \qquad (2.3)$$

$$\text{where} \quad \left.\begin{aligned} G(x) &= a^x g(a^x) = sg(s) \\ K(x) &= \log aa^x e^{-a^x} = \log ase^{-s} \\ F(y) &= f(a^{-y}) = f(t) \end{aligned}\right\} \qquad (2.4)$$

Equation (2.3) occurs widely in the applied sciences. *K* and *G* are known kernel and data functions respectively, and *F* is to be found. It is convenient to assume that the functions *F*, *G* and *K* belong to some suitable function space, such as $L_2(R)$, so that their Fourier transforms (FTs) exist. ($\wedge$ denotes FTs and $\vee$ denotes inverse FTs).

The ill-posedness of (2.3) is well-known; the degree of ill-posedness depends on the decay rate of the kernel transform $\hat{K}(\omega)$ and also on the degree of smoothness of the solution F. We may seek a stable or filtered approximation to F given by

$$F_\lambda(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega;\lambda)\, \frac{\hat{G}(\omega)}{\hat{K}(\omega)} \exp(i\omega y)\, d\omega \qquad (2.5)$$

where $Z(\omega; \lambda)$ is a stabilizing or filter function dependent on a parameter λ.

Filters may be constructed in several ways, in particular from Tikhonov regularization methods where λ in (2.5) becomes the regularization parameter. In the latter case, two essential decisions are needed (i) the choice of smoothing norm which acts as the stabilizing functional and (ii) the choice of λ in some optimal sense.

REGULARIZATION AND APPROXIMATION

We shall restrict our attention to p-th order regularization, where the smoothing functional

$$C(F;\lambda) = \|K * F - G\|_2^2 + \lambda \|F^{(p)}\|_2^2 \qquad (3.1)$$

is minimized over the subspace $H^p \subset L_2$. Both norms in (3.1) are L_2 norms, and $F^{(p)}$ denotes the p-th derivative of F. The regularization parameter λ controls the trade-off between smoothness imposed on the filtered solution and the extent to which (2.3) is satisfied. The minimizer of (3.1) in H^p is given by (2.5) where

$$Z(\omega;\lambda)=\frac{\left|\hat{K}(\omega)\right|^2}{\left|\hat{K}(\omega)\right|^2+\lambda\omega^{2p}} \tag{3.2}$$

We assume throughout this paper that the support of each function F, G and K is essentially finite and contained within the interval [0, *T*], where *T* is the period and *T* = *N* h, *N* is the number of data points and h is the spacing. It is then convenient to adopt the approximating function space T_N, of trigonometric polynomials of degree at most *N* and period *T*, since then the discretization error in the convolution may be made exactly zero at the grid points (see section 5). Fast Fourier transforms (FFT) may be employed in the solution procedure.

Let *G* and *K* be given at *N* equally spaced points $x_n = nh$, n = 0,1 2,…*N*-1, with spacing h = *T/N*. Then *G* and *K* are interpolated by G_N and $K_N \in T_N$ where

$$G_N(x) = \frac{1}{N}\sum_{q=0}^{N-1}\hat{G}_{N,q}\exp(i\omega_q x) \tag{3.3}$$

$$\hat{G}_{N,q} \sum_{n=0}^{N-1} G_n \exp(-i\omega_q x_n) \tag{3.4}$$

where $G(x_n) = G_n = G_N(x_n)$, $\omega_q = \dfrac{2\pi q}{T}$ (3.5)

Similar expressions as (3.3) and (3.4) can be obtained for K_N.

It can be established that:

(1) Equation (2.3) is exactly equivalent at (x_n) to the $N \times N$ discrete systems

$$(KF) = G_n, \qquad n = 0,1,\ldots, N-1 \tag{3.6}$$

where $K = \psi \operatorname{diag}(\hat{K}_{N,q})\psi^H$ (3.7)

and ψ is the unitary discrete FT matrix with elements

$$\psi_{rs} = N^{-\frac{1}{2}} \exp\left(\frac{2\pi i}{N} rs\right),\quad r,s = 0, 1,\ldots, N-1 \tag{3.8}$$

(2) In T_N, F_λ in (2.5) is approximated by

$$F_{N,\lambda}(x) = \sum_{q=0}^{N-1} Z_{q;\lambda} \frac{\hat{G}_{N,q}}{\hat{K}_{N,q}} \exp(i\omega_q x) \tag{3.9}$$

where the discrete p-th order filter is

$$Z_{q;\lambda} = \frac{\left|\hat{K}_{N,q}\right|^2}{\left|\hat{K}_{N,q}\right|^2 + N^2\lambda\tilde{\omega}_q^{2p}}, \quad \tilde{\omega}_q = \begin{cases} \omega_q, 0 \le q < \dfrac{N}{2} \\ \omega_{N-q}, \dfrac{N}{2} \le q \le N-1 \end{cases} \tag{3.10}$$

DETERMINATION OF λ BY CROSS-VALIDATION METHOD [Talbot 1979]

If the m-th data point G_m is ignored, define the corresponding filtered solution (3.9) by

$$F_{N;\lambda}^{[m]} = \min_{\phi \in T_N} \frac{1}{N} \sum_{\substack{n=0 \\ n \ne m}}^{N-1} \left|(K\phi)(x_n) - G_n\right|^2 + \lambda \left\|\phi^{(p)}\right\|_2^2 .$$

The value (K $F_{N;\lambda}^{[m]}$ (x_m) then predicts the missing datum G_m. The optimal λ is defined as that which minimizes the weight mean-square prediction error for all m:

$$V_{cv}(\lambda) = \frac{1}{N}\sum_{m=0}^{N-1} [(K F_{N;\lambda}^{[m]})(x_m) - G_m]^2 w_m(\lambda). \tag{4.1}$$

For the circulant kernel (3.7) unit weights are appropriate to be applied and (4.1) may then be expressed as

$$V_{cv}(\lambda) = \frac{\dfrac{1}{N}\displaystyle\sum_{q=0}^{N-1} (1 - Z_{q;\lambda})^2 \left|\hat{G}_{N,q}\right|^2}{\left[1 - \dfrac{1}{N}\displaystyle\sum_{q=0}^{N-1} Z_{q;\lambda}\right]^2} \tag{4.2}$$

Equation (4.2) can be minimized with respect to λ. In order to minimize V(λ) in equation (4.2), we have used a subroutine which uses a quadratic interpolation technique to obtain a minimum.

LEMMA

Let $F \in T_N$, i.e. $F(x) = \dfrac{1}{N}\displaystyle\sum_{r=o}^{N-1} \hat{F}_{N,r} e^{i\omega_r x}$ and $F = (F(x_o), F(x_1), \dots, F(x_{N-1}))^T$.

Then the $N \times N$ matrix of the form

$$K = \psi \,\text{diag}\,(\hat{K}_{N,q})\psi^H \tag{5.1}$$

where ψ is the unitary matrix with elements

$$\psi_{rs} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i r s}{N}\right), \qquad r, s = 0,1,2,\dots, N-1$$

satisfies

$$(KF)_n = (K_N * F)(x_n), \qquad n = 0,1,\ldots, N-1. \qquad (5.2)$$

Proof

$$(K_N * F)(x_n) = \int_0^T K_N(x_n - y)\, F(y)\, dy$$

$$= \frac{1}{N}\int_0^T \sum_{q=0}^{N-1} \hat{K}_{N,q} \exp(i\omega_q (x_n - y))\, F(y)\, dy$$

$$= \frac{1}{N^2}\sum_{q=0}^{N-1}\sum_{r=0}^{N-1}\int_0^T \exp(i(\omega_r - \omega_q)\, y)\, dy$$

$$\hat{K}_{N,q}\hat{F}_{N,r} \exp(i\omega_q x_n)$$

using $$\int_0^T e^{i\frac{2\pi}{T}(r-q)y}\, dy = \begin{cases} 0, & r \neq q \\ 1, & r = q. \end{cases}$$

$$= \frac{1}{N}\sum_{q=0}^{N-1} \hat{K}_{N,q}\hat{F}_{N,q} \exp(i\omega_q x_n)$$

$$= \sum_{m=0}^{N-1}\left[\sum_{q=0}^{N-1} \psi_{qn}(\hat{K}_{N,q})\overline{\psi}_{qm}\right] F_m$$

$$= (KF)_n.$$

NUMERICAL RESULTS

In this section we tabulate the results of the above method applied to the test examples taken from Rodriguez and Seatzu [1993]. All data functions have the property $g(s) = 0(s^{-1})$ and no noise is added apart from machine rounding error, only optimal results have been quoted in the table and demonstrated in the diagrams. In each of the test examples, 64 sample points are used to calculate the discrete Fourier coefficients of G and K (Eq.(3.4)).

In our numerical calculations we need to choose two numbers x_{min}, x_{max} such that $|G(x)| < \in$, whenever $x > x_{min}$ and $x < x_{max}$. In what follows we choose $\in = 10^{-4} \max |G(x)|$.

We find x_{min} and x_{max} as the smallest and largest solution of the non-linear equation $G(x) = \in$. We may then pose the deconvolution problem (2.3) on the interval $[0,T]$, where $T = x_{max} - x_{min}$. Since the size of the essential support of G (x) depends upon 'a' (see (2.1) and (2.4)), we have for a fixed number N of equidistant data points $\{x_n\}$, $h = T/N$. We have minimized (4.2) with respect to λ for values of $a \geq 1$ and compared the L_∞ error of the resulting solution with the values of the true solutions. We found that minimum value of V (λ) for optimal λ for which the L_∞ error of the regularized solution is the least.

Example 1: [Rodriguez and Seatzu 1993].

$$g(s) = \exp(-4\sqrt{s})$$

$$f(t) = \frac{2\exp(-4/t)}{\sqrt{\pi t^3}}$$

The optimal results compared with Rodriguez and Seatzu [1993] solution are shown in Figs. 1, 2 and Table 1.

Table 1: Calculation of various parameters

Example No.	a	T	h	λ	V(λ)	$\left\Vert f - f_\lambda \right\Vert_\infty$	Figs. No.
1	10.0	12.50	0.19531	0.21×10^{-12}	0.3461×10^{-12}	1.00×10^{-5}	1
2	05.0	11.50	0.17969	0.18×10^{-11}	0.4083×10^{-8}	0.45×10^{-2}	3
3	10.0	14.50	0.22656	0.11×10^{-12}	0.1881×10^{-7}	0.20×10^{-2}	5

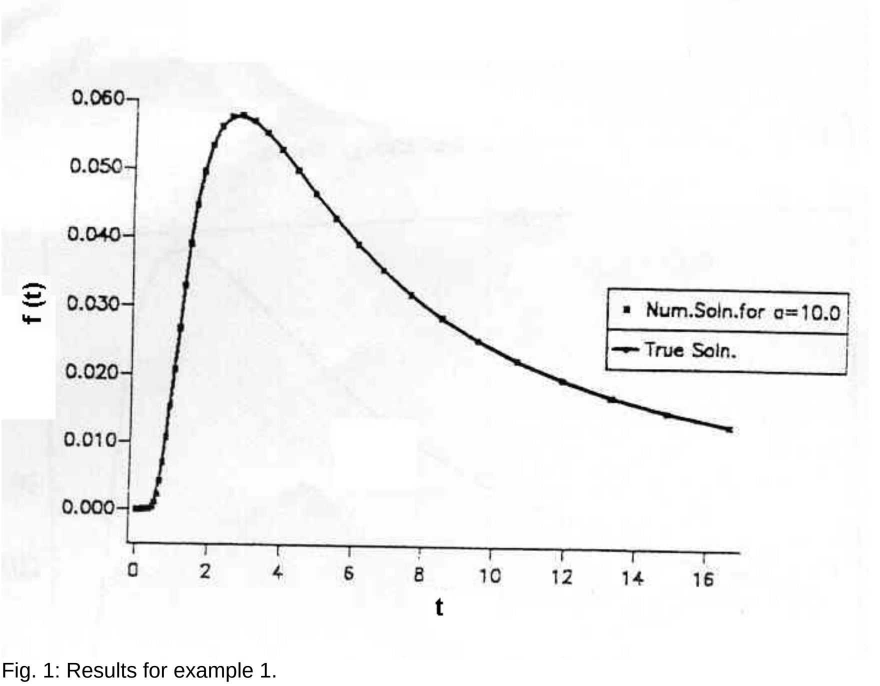

Fig. 1: Results for example 1.

Example 2: [Rodriguez and Seatzu 1993].

$$g(s) = \exp(-\sqrt{s}/2)$$

$$f(t) = \frac{\exp(-1/16t}{\sqrt{16\pi t^3}}$$

The optimal results compared with Rodriguez and Seatzu [1993] solution are shown in Figs. 3, 4 and Table 1.

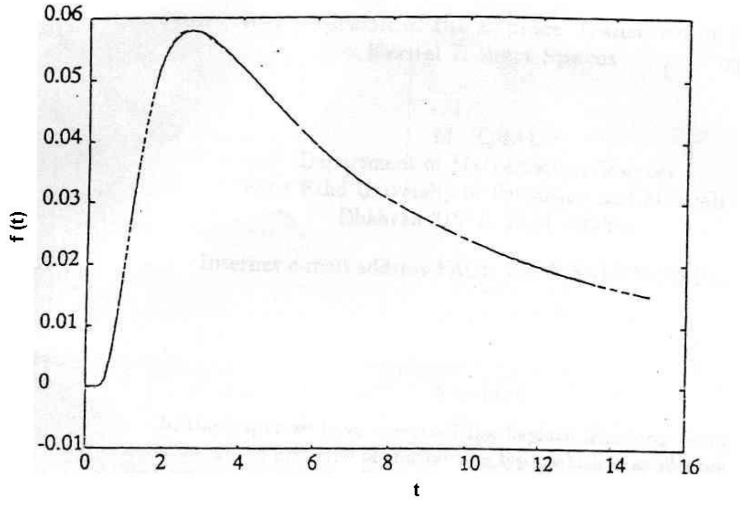

Fig. 2: Results for example 7 of Rodriguez and Seatzu [1993]: test function (solid line), recovery (dashed line).

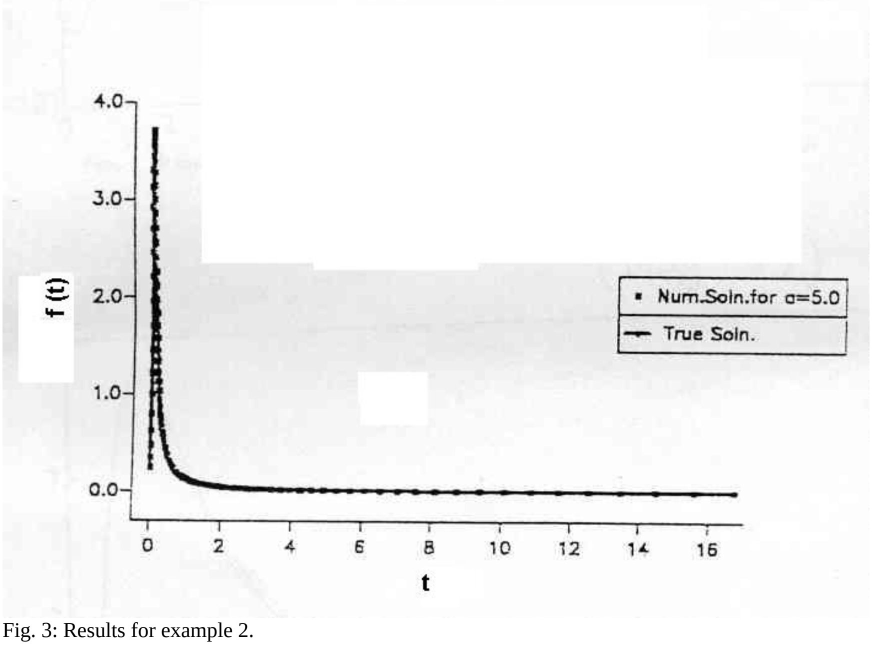

Fig. 3: Results for example 2.

Example 3: [Rodriguez and Seatzu 1993].

$$g(s) = \frac{6}{(s+1)^4}$$

$$f(t) = t^3 \exp(-t)$$

The optimal result is shown in Fig. 5 and Table 1. Rodriguez figure is not available to compare with.

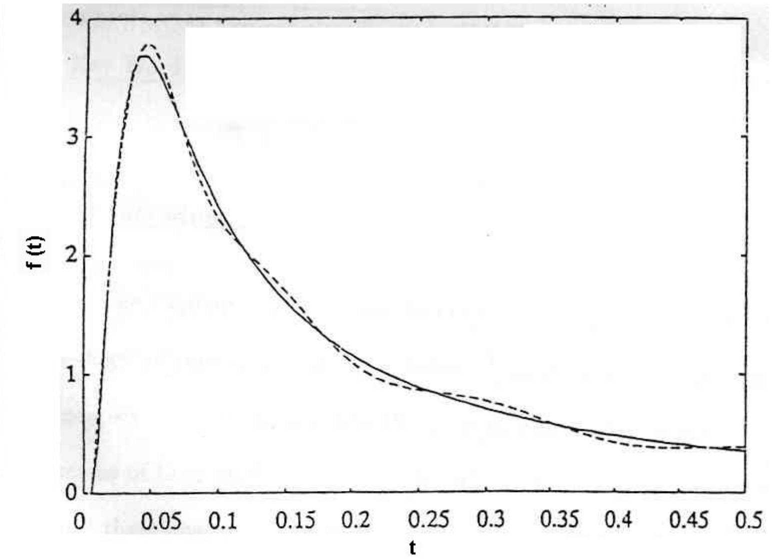

Fig. 4: Results for example 8 of Rodriguez and Seatzu [1993]: test function (solid line), recovery (dashed line).

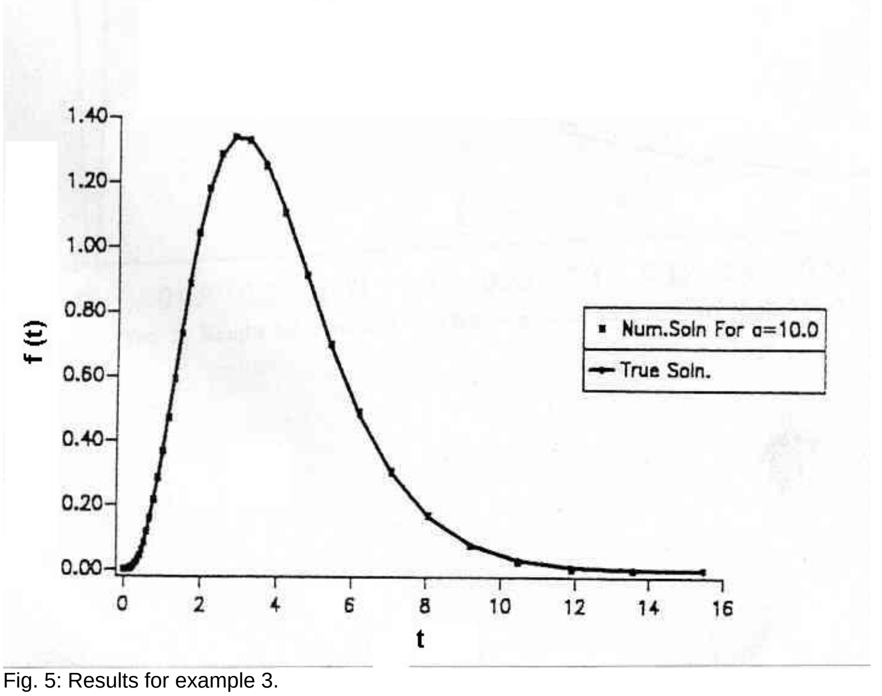

Fig. 5: Results for example 3.

CONCLUSION

The method worked very well over all the three test examples and results obtained are very good as shown in Figs. 1, 3 and 5. As regards comparison with Rodriguez and Seatzu [1993], we have obtained good results and over a wider range of t.

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