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SOME RESULTS ON GREEN'S POLYNOMIALS

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Abstract: The transition matrix X(t) between the power sum product and Hall-Littlewood function gives us Green's polynomials. These polynomials are useful in the construction of the irreducible characters of GL(n, q), see [Green 1955] and [Morris 1963]. In this paper we consider the special case $t = \omega$, where ω is a primitive pth root of unity. Our intension is to give proof of some conjectures [Morris and Sultana 1991].

Keywords: Green's polynomials, partitions, Transition matrix, power sum bases.

INTRODUCTION

We shall first briefly review some of the basic definitions and results. We follow closely the notations used in Macdonald [1995].

Let $\lambda = (\lambda_1, \ldots \lambda_m)$ be a partition of n; that is $\lambda_1 + \ldots + \lambda_m = n$, $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_m > 0$. Length of the partition is denoted by $\lambda(\lambda) = m$. We shall write $|\lambda|$ for $\Sigma\lambda_i$, called weight of the partition.

We some time write a partition λ in the form

$$\lambda = (1^{m_1}\ 2^{m_2}\ \ldots\ r^{m_r}).$$

where $m_i = m_i(\lambda)$ is the number of parts of λ which are equal to i.

Let @(n) denotes the set of all partition of n. A partition λ of n is row λ-singular if $\lambda_{i+1} = \lambda_{i+2} = \ldots = \lambda_{I+\lambda} > 0$ for some i. Otherwise λ is row λ-regular.

A partition λ of n is row λ-singular if λI λ_I for some i. Otherwise λ is column λ-singular.

For later use we let $@_\lambda$(n) and $@^\lambda$(n) denote the set of row λ-regular and column λ-regular partition of n respectively. Then it is well known that $|@_\lambda(n)| = |?^\lambda(n)|$ [James and Kerber 1981].

Let $x_1, x_2, \ldots$ be an infinite set of indeterminates and t an indeterminate independent of the $x_1, x_2 \ldots x_r$ Let P_λ (x; t) and $Q_\lambda(x_n; t)$ be the Hall-Littlewood P and Q-functions defined by [Macdonald 1995]. We may have

$$Q_\lambda(x; t) = b_\lambda(t)\, P_\lambda(x; t) \tag{1.1}$$

where

$$b_\lambda(t) = \prod \phi_{m_i(\lambda)}(t) \tag{1.2}$$

and

$$\phi_r(t) = (1 - t)(1 - t^2) \ldots (1 - t^r) \tag{1.3}$$

Then we define another function

$$Q_r(x; t) = q_r(x; t)$$

Moreover by using differential operator on Hall-Littlewood P and Q-functions it has shown in [Sultana 1999] that

$$n \frac{\partial}{\partial p_n} q_N = (1 - t^n)\, q_{N-n} \qquad (1.4)$$

where p_i is power sum symmetric function $p_i = \sum_{r=1}^{\infty} x_r^i$ (i = 1,2,…)

The transition matrix X (t) connecting the P_λ (x; t) or Q_λ (x_n; t) with power-sum bases $\{p_\rho(x) = p_1^{\rho_1} \ldots p_n^{p_n}\}|\quad \rho = (1^{\rho_1} \ldots n^{p_n})$ is called Green's polynomial, that is

$$p_\rho(x) = \Sigma\ X_\rho^\lambda(t)\ P_\lambda(x; t) \qquad (1.5)$$

and

$$Q_\lambda(x; t) = \Sigma\ z_\rho(t)^{-1}\ X_\rho^\lambda(t)\ p_\lambda(n) \qquad (1.6)$$

where z_ρ (t) = $\prod_{j \ge 1} \frac{(1-t^{\rho_i})}{i^{\rho_i}\ \rho_i!}$ and thus X_ρ^λ (t) $\in$ Z[t] [Sultana 1999].

The following is a generalization of the Murnaghan-Nakayama recursive formula in [Murnaghan 1938] for calculating irreducible characters of symmetric group S_n. We give here a proof of this result.

Theorem

Let X_ρ^λ (t) be the Green's polynomial and $\rho' = (1^{\rho_1}\ 2^{\rho_2} \ldots n^{\rho_{n-1}} \ldots N^{\rho_N})$, then $X_\rho^\lambda(t) = \sum_{\mu \in M(\lambda)} X_{\rho'}^\mu(t)$, where M(λ) is the set of m partitions μ of *N-n* obtained by subtracting n from each part of λ in turn.

Proof

By theorem given by Macdonald [1995] on page 107 we have

$$Q_\lambda(x; t) = \prod_{i \ge 1}\ 1 + (t-1)\, R_{ij} + (t^2 - t)\, R_{ij}^2 + \ldots)\ q_\lambda(x; t)$$

Where R_{ij} is the operator, which replaces λ_i by λ_{i+1} and λ_j by λ_{j-1}. Thus

$$Q_\lambda(x; t) = \sum\ \psi_R(t)\ q_{R\lambda},$$

where $\psi_R(t) \in Z[t]$

we see that $q_{R\lambda} = 0$ if any part of R_λ is negative. If we replace λ by $\lambda - n\in_i$, where $\in_i$ = (0, …, 0, 1, 0, …,0) with 1 in the ith place we obtain

$$\sum_{i=1}^{m} Q_{\lambda - n\in_i} = \sum_R\ \psi_R(t) \sum_{i=1}^{m} q_{R(\lambda - n\in)},$$

or

$$(1 - t^n) \sum_{i=1}^{m} Q_{\lambda - n\in_i} = \sum_{R} \psi_R(t) \sum_{i=1}^{m} (1 - t^n)\, q_{R(\lambda - n\in_i)}$$

Now by using (1.4) we have

$$(1 - t^n) \sum_{i=1}^{m} Q_{\lambda - n\in_i} = \Sigma\, \psi_R(t) \sum_{i=1}^{m} q_{(R\lambda)_1} \cdots n \frac{\partial}{\partial p_n} q_{(R\lambda)_i} \cdots q_{(R\lambda)_{rn}}$$

$$= n \frac{\partial}{\partial p_n} Q_\mu,$$

Therefore, $$n \frac{\partial}{\partial p_n} Q_\lambda(x; t) = (1 - t^n) \sum_{\mu \in M(\lambda)} Q_\mu(x; t) \tag{1.7}$$

Now substituting (1.6) in (1.7) and then comparing the coefficients of $p_1^{\rho_1} \ldots p_n^{\rho_n - 1} \ldots p_N^{\rho_N}$. We get the required result.

Also for t = 0 since $P_\lambda(x; 0) = S_\lambda$, where S_λ is Schur function defined by [Schur 1911], we have

$$p_\rho = \Sigma\, X_\rho^\lambda S_\lambda \quad \text{since} \quad X_\rho^\lambda(0) = \chi_\rho^\lambda$$

where χ_ρ^λ is the value of irreducible character χ^λ of S_n at elements of cycle-type ρ.

For the case t = - 1. If $\lambda \in @_2(n)$ and $\rho \in @^2(n)$, then

$$X_\lambda^\rho(-1) = 2^{(\lambda(\lambda) - \lambda(\rho) + \in(\lambda))/2} \xi_\rho^\lambda$$

where ξ_ρ^λ are the irreducible spin characters of S_n and $\in(\lambda)$ = 1 or 0 according as n - $\lambda(\lambda)$ is odd or even.

From Macdonald [1995] we have the following results:

$$X_n^\lambda(t) = t^{n(\lambda)} \phi_{\lambda(\lambda)-1}(t^{-1}) \tag{1.8}$$

where $n(\lambda) = \Sigma(i - 1)\lambda_i$ and $\phi_r(t) = (1 - t) \ldots (1 - t^r)$

and $$X_\rho^{1^n}(t) = \prod_{i=1}^{n} (t^{i} - 1) / \prod_{j \geq 1} (t^{\rho_{j-1}}) \tag{1.9}$$

RESULTS ON X(ω) WHERE ω IS PRIMITIVE pth ROOT OF UNITY

Now we shall discuss the transition matrix X(ω) which is special case of X(t) where t = ω, ω is a primitive pth root of unity. Then X(ω) are of great interest in the modular representation theory. Morris and Sultana [Morris and Sultana 1991] discussed the transition matrix $X(\omega)$ for n = 3,4,5 and for p = 2, 3.

We now divide the matrix $X(\omega)$ into four submatrices say $X_1(\omega)$, $X_2(\omega)$, $X_3(\omega)$ and $X_4(\omega)$ arranged as follow.

$$\begin{array}{ll|cc|}
 & & @_\rho(n) & @(n)/ \\
@_\rho(n) & & & \\
 & @^\rho(n) & X_1(\omega) & X_2(\omega) \\
 & @(n)/@^\rho(n) & X_3(\omega) & X_4(\omega)
\end{array}$$

It was noted that $X_4(\omega)$ are either zero or divisible by p.

There are two conjectives given by Sultana [1990], in this section we shall prove those conjectives for some partitions. By using (1.8) and (1.9) we shall prove the following results.

Theorem

Let λ = kp, where p is any prime, then $X_n^{(k^p)}(\omega) = p\,\omega^{\frac{(k-1)(p-1)}{2}}$.

Proof

Since by (1.7) we have

$$X_n^{\lambda}(t) = t^{n(\lambda)}\phi_{\lambda(\lambda)-1}(t^{-1}) .$$

Thus for t = ω, ω is primitive pth root of unity and $\lambda = k^p$, we have

$$n(\lambda) = \sum_{i=1}^{p} (i-1)k$$

$$= \sum_{i=1}^{p-1} ik.$$

and

$$\phi_{\lambda(\lambda)-1}(\omega^{-1}) = \prod_{i=1}^{p-1} (1 - \omega^{-1})^i$$

$$= \omega^{-\sum_{i=1}^{p-1} i} \prod_{i=1}^{p-1} (\omega^i - 1)$$

$$= p\ \omega^{-\sum_{i=1}^{p-1} i}$$

Hence

$$\chi_n^{(k^p)}(\omega) = \text{p } \omega^{-\sum_{i=1}^{p-1} i} \quad \omega^{k\sum_{i=1}^{p-1} i}$$

$$= \text{p } \omega^{(k-1)} {}^{-\sum_{i=1}^{p-1} i}$$

$$= \text{p } \omega^{\frac{(k-1)(p-1)}{2}}$$

which is required result.

Remark

Above theorem shows that

$$\text{p}/X_n^{k^p}(\omega).$$

Theorem

Let n = kp + j, where p is primitive pth root of unity and j < p. Then

$$X_\rho^{(1^n)}(\omega) = k! \quad p^k \quad \tilde{X}_\mu^{(1^j)}(\omega) \qquad \text{whenever } \rho \text{ has k parts equal to p}$$
$$= 0 \text{ otherwise,}$$

where $\tilde{X}_\mu^{(1^j)}(\omega)$ is the polynomial for the transition matrix in which rows and columns are indexed by the partitions of j.

Proof

Let ρ = (k^ρ, $\mu_1,\ldots,u_r$), where ($\mu_1,\ldots,u_r$) be any partition of k. Now in equation (1.8) for t = ω, where ω is primitive pth root of unity. We get

$$\tilde{X}_\mu^{(1^j)}(\omega) = \prod_{i=1}^{kp+j} (\omega^i - 1) / \prod_{j\geq 1} (\omega^{\rho_j} - 1)$$

$$= \text{p}^k \text{ k! } \frac{\prod_{i=1}^{j}(\omega^j - 1)}{\prod_{j=1}^{r}(\omega^{\mu_j} - 1)}$$

$$= \text{p}^k \text{ k! } \quad \tilde{X}_\mu^{(1^j)}(\omega).$$

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