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NUMBER OF 2π-PERIODIC SOLUTIONS OF THE NON-AUTONOMOUS EQUATION IN THE WHOLE PLANE

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Abstract: 2π-periodic solutions of certain one dimensional non-autonomous cubic differential equations are investigated in the whole plane when the coefficient of cubic term in the equation does not change sign, using the transformations (1.3). Also we have proved that the transformation can be applied to non-homogeneous system (4.1) for possible investigation of 2π periodic solutions.

Keywords: 2π-Periodic Solutions, Limit Cycles, Stable, Unstable.

INTRODUCTION

We consider the differential systems of the form

$$\begin{aligned} \dot{x} &= \lambda x + y + p_n(x,y) \\ \dot{y} &= -x + \lambda y + q_n(x,y) \end{aligned} \qquad (1.1)$$

where p_n and q_n are homogeneous polynomials in x, y of degree n and λ is a real constant. In polar form equation (1.1) can be written as

$$\begin{aligned} \dot{r} &= \lambda r + r^n f(\theta) \\ \dot{\theta} &= -1 + r^{n-1} g(\theta) \end{aligned} \qquad (1.2)$$

where f(θ)and g(θ) are homogeneous polynomials of degree n + 1 in cosθ and sin θ. Now we define

$$\rho = r^{n-1} (1 - r^{n-1} g(\theta))^{-1} \qquad (1.3)$$

Then after some manipulation, we obtain

$$\frac{d\rho}{d\theta} = \alpha(\theta)\rho^3 - \beta(\theta)\rho^2 - \lambda(n-1)\rho \qquad (1.4)$$

where

$$\alpha(\theta) = -(n-1)g(\theta) \{f(\theta) + \lambda g(\theta)\}$$

$$\beta(\theta) = -(n-1)\{f(\theta) + 2\lambda g(\theta)\} + g'(\theta)$$

are homogenous polynomials in cosθ and sinθ of degree 2n+2 and n+1 respectively. The transformation (1.3) is defined in an open set D containing the origin, where

$$D = \{r,\theta);\ r^{n-1} g(\theta) < 1\}, \qquad (1.5)$$

In this paper we are mainly concerned with the number of 2π-periodic solution of (1.4) or the number of limit cycles of (1.1) in the whole plane when the associated function α(θ) in (1.4) does not change sign. Previously results have been obtained about the number of 2π-periodic solutions of (1.4) inside the set D [Cherkas 1976, Lloyd 1982]. We were motivated by the paper of Carbonell and Llibre [1997]. The relationship between (1.1) and (1.3) will be explained in section 2. In section 3 results will be obtained for homogeneous systems (1.1). In section 4 we will show how the transformation (1.3) can be applied to non-homogeneous systems to investigate the number of 2π-periodic solutions.

THE TRANSFORMATION

The transformation (1.3) is used to investigate the number of limit cycles of (1.1). For Quadratic systems see Coppel [1986] and [Lins Neto 1980] D extends at least as far as the outermost limit cycles surrounding the origin. Note that the transformation (1.3) maps $r = 0$ to $\rho = 0$, $r > 0$ to $\rho > 0$ and the neighbourhood of $r = 0$ to neighbourhood of $\rho = 0$. If $\rho > 0$ and $1 + g(\theta) > 0$ for all θ, we invert (1.3), then r is the positive (n-1) the root of

$$\frac{\rho}{1+\rho g(\theta)} \qquad (2.1)$$

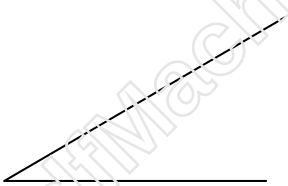
It is easy to check that for system (1.1) the critical point at the origin corresponds to the constant solution $\rho = 0$ of (1.4) and limit cycles of (1.1) correspond to 2π-periodic solution of (1.4) with ρ small and positive.

LIMIT CYCLES

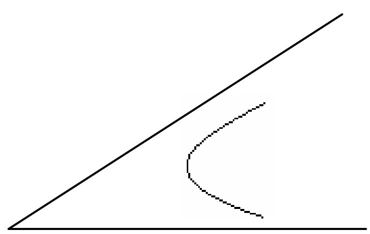
To investigate the number of limit cycles of (1.1) or 2π-periodic solution of (1.4) when $\alpha(\theta)$ in (1.4) does not change sign. We define curve.

$$K = \{r, \theta): \dot{\theta} = 0\} \qquad (3.1)$$

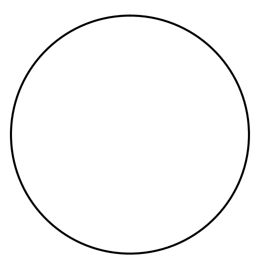
Then on K, $r^{n-1} = \dfrac{1}{g(\theta)}$. The possible forms of K when n is odd are as follows



$g(\theta_i) = 0 \quad i = 1,2$
$g(\theta) < 0 \quad$ for all $\theta \in (\theta_1\ \theta_2)$



$g(\theta_i) = 0 \quad i = 1,2$
$g(\theta) > 0 \quad$ for all $\theta \in (\theta_1\ \theta_2)$



n is odd and
$g(\theta) > 0 \quad$ for all θ.

We are interested in the case (iii) when n is odd and $g(\theta) > 0$ for all θ. In this case K is simple closed curve which divides the plane into two parts; $\dot{\theta} < 0$ and $\dot{\theta} > 0$. To see the number of limit cycles inside the curve K, we use the transformation (1.3). We suppose that $\alpha(\theta)$ in (1.4) never zero. It is easily seen that the constant solution $\rho = 0$ and $\rho = -\dfrac{1}{g(\theta)}$ if $g(\theta) > 0$ for all θ are 2π-periodic solutions of (1.4).

To investigate the number of limit cycles outside the curve K, we use the transformation

$$\sigma = \frac{1}{r^{n-1} g(\theta) - 1} \tag{3.2}$$

If $\sigma > 0$, we can invert (3.2) and we have

$$r^{n-1} = \frac{\sigma + 1}{\sigma g(\theta)} \tag{3.3}$$

Then in (σ,θ) co-ordinates, system (1.1) becomes

$$\frac{d\sigma}{d\theta} = A(\theta)\sigma^3 + B(\theta)\sigma^2 + C(\theta)\sigma \tag{3.4}$$

where

$$A(\theta) = -(n-1)\{f(\theta) + \lambda g(\theta)\}/g(\theta) = \alpha(\theta)/(g(\theta))^2$$

$$B(\theta) = -(n-1)\{2f(\theta) + 2\lambda g(\theta)\}/g(\theta) - g'(\theta)/g(\theta)$$

$$C(\theta) = -(n-1) f(\theta)/g(\theta) - g'(\theta)/g(\theta)$$

The transformation (3.2) maps $r = 0$ to $\sigma = -1$, $r = \infty$ to $\sigma = 0$ and the curve K into $\sigma = \infty$. It can be seen that $\sigma = 0$ and $\sigma = -1$ are periodic solutions of (3.4). Let

$$S(\sigma,\theta) = A(\theta)\sigma^3 + B(\theta)\sigma^2 + C(\theta)\sigma$$

and we denote the return map associated with this system by $h^*(x)$. Then the derivative of the return map [Lloyd, 1982] at $\sigma = 0$ is

$$h^{*\prime}(x) = \exp[(n-1) D_1] \tag{3.5}$$

where

$$D_1 = -\int_o^{2\pi} \frac{f(\theta)}{g(\theta)}$$

Since $A(\theta) = \dfrac{\alpha(\theta)}{(g(\theta))^2}$ and as $\alpha(\theta)$ is of one sign $g(\theta) > 0$ for all θ; therefore $A(\theta)$ is of one sign. Thus in different cases we have the following results.

Theorem

Let n be odd and $\alpha(\theta) \geq 0$. Then there is one limit cycle if $\lambda\alpha(\theta) \geq 0$ and no limit cycle if $\lambda\alpha(\theta) \leq 0$.

Proof

Since $\alpha(\theta) = -(n-1)\, g(\theta)\, \{f(\theta) + \lambda\, g(\theta)\} \geq 0$. Then $A(\theta) = \dfrac{\alpha(\theta)}{\{g(\theta)\}^2} \geq 0$.

Because $g(\theta) > 0$ for all θ. Let $H(\theta) = f(\theta) + \lambda\, g(\theta)$. Then $\alpha(\theta) \geq 0$ implies $H(\theta) < 0$; as $g(\theta) > 0$ for all θ. On the curve K, $\dot{\theta} = 0$ and $\dot{r} = r\, H(\theta)/g(\theta)$; therefore the vector field on K points radially inward. Now

(i) **<u>Let $\lambda > 0$.</u>** Then from (1.1) it is clear that the origin is unstable, thus by Poincare Bendixon theorem there is at least one closed orbit between $\rho = 0$ and the curve K. To look at infinity, we use the derivative of return map given in (3.5). Since $H(\theta) < 0$ and $\lambda > 0$, $g(\theta) > 0$. Therefore $f(\theta) \leq 0$ and hence (3.5) gives $D_1 > 0$; that is $\sigma = 0$ or infinity is unstable (repelling). So we have at most two limit cycles between $\sigma = 0$ and K or non. As the maximum number is three, and two namely $\sigma = 0$ and $\sigma = -1$ are known, therefore if $\lambda > 0$ and $\alpha(\theta) \geq 0$ there exists exactly one limit cycle inside K.

(ii) **<u>Let $\lambda < 0$</u>.** Then $H(\theta) = f(\theta) + \lambda\, g(\theta) \leq 0$ implies that either $f(\theta) \leq 0$ or $f(\theta) \geq 0$. If $g(\theta) \leq 0$ then $D_1 > 0$ and hence by (3.5) $\sigma = 0$ is unstable. So there is no limit cycle between the curve K and $\sigma = 0$. Since $\lambda < 0$ then the origin ($\rho = 0$) is stable and the vector field is inward on K, therefore there are two limit cycles or no limit cycle between $\rho = 0$ and the curve K. Since $\rho = 0$ and $\rho = -\dfrac{1}{g(\theta)}$ are known and there are at most three limit cycles in total, thus there is no limit cycle between $\rho = 0$ and K if $\lambda < 0$ and $\alpha(\theta) \geq 0$. Similarly if $f(\theta) \geq 0$ then $D_1 \leq 0$ and $\sigma = 0$ is stable; therefore by Poincare - Bendixon theorem there exists one limit cycle between $\sigma = 0$ and K and non between $\rho = 0$ and K.

Now let $\alpha(\theta) \leq 0$; then the vector field on the curve K is radially outward. Following the same procedure as above, we have the following result.

Theorem

Let n be odd and $\alpha(\theta) \leq 0$. Then there is no limit cycle if $\lambda\alpha(\theta) \leq 0$ and there is one limit cycle if $\lambda\alpha(\theta) \geq 0$.

EXTENSION TO NON-HOMOGENEOUS SYSTEM

In the previous papers, work has been done for the systems of the form (1.1). In this section we show how the transformation described in section 2 can be applied to the systems of the form

$$\dot{x} = \lambda x + y + P_n(x, y) + x R_{m-1}(x, y)$$

$$\dot{y} = -x + \lambda y + q_n(x, y) + y R_{m-1}(x, y) \qquad (4.1)$$

where p_n and q_n are homogeneous polynomials in x and y of degree n and R_{m-1} is a homogeneous polynomial of degree m-1 in x and y.
In polar co-ordinates the system (4.1) can be written as

$$\dot{r} = \lambda\gamma + r^n f(\theta) + r^m R(\theta)$$
$$\dot{\theta} = -1 + r^{n-1} g(\theta) \qquad (4.2)$$

where

$$f(\theta) = \cos\theta\, p_n(\cos\theta, \sin\theta) + \sin\theta\, q_n(\cos\theta, \sin\theta)$$
$$g(\theta) = \cos\theta\, q_n(\cos\theta, \sin\theta) - \sin\theta\, p_n(\cos\theta, \sin\theta)$$
$$R(\theta) = R_{m-1}(\cos\theta, \sin\theta)$$

are homogeneous polynomials of degree n + 1, n + 1 and m – 1 respectively. Since $\dot{\theta}$ in (4.2) is independent of R(θ), we can use the transformation (1.3). After some manipulation, in (ρ,θ) co-ordinates we have

$$-\frac{d\rho}{d\theta} = (n-1)\lambda\rho + \rho^2(n-1)(f(\theta) + 2\lambda g(\theta)) - g'(\theta)$$
$$+ \rho^3\{(n-1)(f(\theta)g(\theta) + \lambda(g(\theta))^2\} + \frac{(n-1)r^{m+n-1}R(\theta)}{\{1-r^{n-1}g(\theta)\}^3} \qquad (4.3)$$

From (4.3) it is clear that we can only transform the system (4.1) into the form (1.4) if m = 2n –1, then the system (4.1) in (ρ,θ) co-ordinates becomes

$$\frac{d\rho}{d\theta} = \alpha(\theta)\rho^3 + \beta(\theta)\rho^2 - (n-1)\lambda\rho \qquad (4.4)$$

where

$$\alpha(\theta) = -(n-1)\{g(\theta)(f(\theta) + \lambda g(\theta)) + R(\theta)\}$$
$$\beta(\theta) = -(n-1)\{f(\theta) + 2\lambda g(\theta)\} + g'(\theta)$$

are homogeneous polynomials in sinθ and cosθ of degree 2n+2 and n+1 respectively. Note that R(θ) is of degree 2n-2, we can make it of degree 2n+2 multiplying by $(\cos^2\theta + \sin^2\theta)^2$. Thus the only systems which are amenable to the transformation (1.3) are of the form (4.1) with m = 2n – 1. Our aim in this section is to consider equation (4.4) when α(θ) does not change sign and to explore some of the consequences for the corresponding system (4.1). We therefore suppose that α(θ) and β(θ) are as given in (4.4). With regard to the system (4.1), the critical point at the origin of (4.1) corresponds to the constant solution ρ = 0 of (4.4) and limit cycles of (4.1) corresponds to 2π-periodic solution of (4.4) with ρ > 0.
Let I be an interval on the positive x-axis, ρ > 0. Let ρ(θ,x) denote the solution of (4.5) satisfying ρ(0,x) = x. If ρ (2π,x) exists we define the return map h(x) =ρ (2π,x). Then 2π-periodic solution of (4.5) corresponds to the zeros of the displacement function H(x) = h(x) – x. To obtain the

results about limit cycles of system (4.1) we shall use the following proposition [Gasull *et al.* 1987].

PROPOSITION

Assume that α does not change sign and $\alpha(\theta) \neq 0$. Let h(x) be the return map associated with the system (4.4). Then the equation h(x) = x has at most three roots, counting multiplicity.

PROPOSITION [Lloyd 1997]

If α does not change sign and $\alpha(\theta) \neq 0$, then system (4.4) has at most three periodic solutions.

To investigate the number of positive 2π-periodic solution, we consider two cases:

(i) When n is odd, (ii) When n is even.

When n is even, $g(\theta)$ and $\beta(\theta)$ are of odd degree; hence $\int_o^{2\pi} \beta(\theta) = 0$.

Moreover if $\rho(\theta)$ is 2π-periodic solution of (4.5). So is $-\rho(\theta + \pi)$, and the two are of opposite sign. Thus we have the following results.

Theorem

Suppose that n is odd and $\alpha(\theta)$ does not change sign. If $\lambda \neq 0$, then there at most two positive 2π-periodic solution; there can be no more than one if $\alpha \geq 0$ and $\lambda > 0$ or $\lambda = 0$.

Proof

Since $\alpha(\theta)$ does not change sign, by proposition (4.2) equation (4.4) can have at most three 2π-periodic solution. But $\rho = 0$ is one of them. Therefore there are at most two positive 2π -periodic solutions. If $\lambda = 0$ and $\alpha(\theta) \geq 0$. Then using derivative of the return map [Lloyd, N.G., 1982], the origin is periodic solution of multiplicity at least two and $h'''(x) > 0$. So there can be at most one positive 2π-periodic solution. If $\alpha(\theta) \geq 0$ and $\lambda > 0$ then $H(x) < 0$ and $H'''(x) > 0$. Thus $h(x) - x$ can not have more than one zero if $\alpha \geq 0$ and $\lambda > 0$.

Theorem

Suppose that n is even and that $\alpha(\theta)$ does not change sign. If $\lambda \neq 0$, there are no positive 2π-periodic solution if $\lambda\alpha \leq 0$ and at most one if $\lambda\alpha \geq 0$. If $\lambda = 0$, there are no non-trivial periodic solution.

Proof

Since $\alpha(\theta)$ does not change sign there are at most three 2π-periodic solution. But $\rho = 0$ is one of them, since n is even, if $\rho(\theta)$ is a 2π-periodic solution of (4.4) then - $\rho(\theta + \pi)$ is also a 2π-periodic solution so there can be only one periodic solution. If $\lambda=0$ then the origin is at least of multiplicity two and in this case periodic solution occur in pairs and they are of different sign.

So there can be no non-trivial periodic solution of either sign. If $\lambda \neq 0$ and $\lambda\alpha(\theta) \leq 0$. Suppose $\alpha \geq 0$ then $\lambda < 0$. Thus $h(x) - x > 0$ for sufficiently small x and $H'''(x) > 0$. Since in this case 2π-periodic solutions occur in pairs therefore $\rho = 0$ is the only periodic solution, so there is no positive 2π-periodic solutions if $\lambda\alpha \geq 0$, suppose $\alpha \geq 0$ then $\lambda > 0$, therefore using the derivative of the return map [Lloyd, 1982]. We have $H(x) < 0$ and $H'''(x) > 0$ so $h(x) - x = H(x)$ can have at most one positive zero, so there is at most one positive 2π-periodic solutions if $\lambda\alpha \geq 0$. The proof is similar if we suppose $\alpha \leq 0$.

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