▼ Journal of Research (Science), Bahauddin Zakariya University, Multan, Pakistan.
Vol. 18, No. 3, July 2007, pp. 177-183 ISSN 1021-1012

SOME RESULTS INVOLVING GENERALIZED PROLATE SPHEROIDAL WAVE FUNCTION AND I-FUNCTION WITH APPLICATION

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Abstract

In this paper, we employ the generalized prolate spheroidal wave function, generalized hypergeometric function, general class of polynomials and the I-function in obtaining the formal solution of a partial differential equation related to a problem of heat conduction in an anisotropic material. This type of problem occurs mainly in wood technology, soil mechanics and the mechanics of solids of fibrous structure. At the end, we give an application of our main finding by inner-connecting them with the Riemann- Liouville type of functional integral operator. The results of this paper unify and extend a large number of results established by various earlier workers.

Keywords: Generalized prolate spheroidal wave function, general class of polynomials, I-function.
(2000 Mathematics subject classification: 33c 90)

INTRODUCTION

One of the solutions of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}+[(\beta-\alpha)-(\alpha+\beta+2)x]\frac{dy}{dx}+[\zeta(s)-s^2x^2]y=0 \qquad (1.1)$$

is the generalized prolate spheroidal wave function [Gupta *et al.* 1985, p. 107, Eq. (2.11)] which denotes as

$$\phi_n^{\alpha,\beta}(s,x)=\sum R_{p,n}^{\alpha,\beta}(s)\,P_{p+n}^{\alpha,\beta}(x) \qquad (1.2)$$

Where $P_n^{\alpha,\beta}$ is the well known Jacobi polynomials and the coefficients $R_{p,n}^{\alpha,\beta}(s)$ satisfy the following relation:

$$R_{n,j}^{\alpha,\beta,n}(s)=\sum_{p=|j|}^{\infty}A_j^p(\alpha,\beta,n)\,s^{2p} \quad j=-n,-n+1,... \qquad (1.3)$$

and $\zeta(s)=n(\alpha+\beta+n+1)+\sum_{j=1}^{\infty}s^{2j}a_j(\alpha,\beta,n)$ (1.4) where $a_j^{'}s$ are independent of s, and $A_j^p(\alpha,\beta,n)$ is defined to be zero if $|j| > 2p$ or j<-1 or j=0 and $p \neq 0$. Besides this we have

$$A_m(\alpha,\beta,n) = 0\ , m \neq 0, A_0(\alpha,\beta,n) = 1,$$

when s$\rightarrow$0 the solution of (1.1) is the Jacobi polynomials of order n. For more details see Gupta *et al*. [1985].

As examples of the application of the generalized hypergeometric function, a general class of polynomials and the I-function, we consider the problem of obtaining solution of a problem of heat conduction.

Consider the partial differential equation related to a problem of heat conduction in an anisotropic material, which has been obtained by Saxena and Nageria [1974] given below:

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] - \frac{pev}{\lambda}\frac{\partial u}{\partial x} + \frac{Q(x)}{\lambda} = \frac{pc}{\lambda}\frac{\partial u}{\partial t} \tag{1.5}$$

With the law of conductivity $K = \lambda(1-x^2)$, Q(x) is the intensity of a continuous source of heat situated inside the solid . Let the initial temperature of the solid to be given by

$$u(x,0) = F(x) \tag{1.6}$$

Taking $v = \dfrac{\alpha-\beta}{q}, Q(x) = -(\alpha+\beta)\lambda x\dfrac{\partial u}{\partial x} - s^2x^2\lambda u, q = \dfrac{pc}{\lambda}$ in (1.5), a solution of (1.5) assume the from

$$U(x,t) = \sum_{l=0}^{\infty} A_l\, e^{-B_l t}\, \phi_l^{\alpha,\beta}\left[s, 1-2\frac{(x-a)}{(b-a)}\right] \tag{1.7}$$

In this paper, we shall assume that

$$F(x) = (x-a)^{\rho-\alpha}(b-x)^{\sigma-\beta} S_n^m\left[y_1(x-a)^{\gamma_1}(b-x)^{\alpha_1}\right] {}_RF_K\left[\begin{matrix}(e_R)\\(f_K)\end{matrix}; y(x-a)^g(b-x)^{\omega}\right] I\left[z(x-a)^h(b-x)^k\right] \tag{1.8}$$

The general class of polynomials defined by Sarivastava [1972] is represented in the following manner:

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k, \text{ n=0,1,2,…} \tag{1.9}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}\ (n,k \geq 0)$ are arbitrary constants, real or complex.

The I-function will be defined and represented as follows [Sexana 1982]:

$$I[Z] = I_{p_i,q_i:r}^{m,n}[Z] = I_{p_i,q_i:r}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_L \phi(\xi)\, z^{\xi}\, d\xi \tag{1.10}$$

where

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}} \tag{1.11}$$

For the convergence, existence conditions and other details of the I-function, we refer to the original paper by Saxena [1982].
We shall use the following notations:

$$A^*=(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \quad ; \quad B^*=(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}$$

The following results will be required here:

$$\int_a^b (x-a)^p(b-x)^\sigma P_n^{\alpha,\beta}\left[1-2\frac{(x-a)}{(b-a)}\right]I\left[z(x-a)^h(b-x)^k\right]dx=\frac{(-n)_m(\alpha+\beta+n+1)_m}{(\alpha+1)_m\, m!}$$
$$I^{m,n+2}_{p_i+2,q_i+1:r}\left[\begin{matrix}(-\sigma,k),(-m-\rho,h),A^*\\ B^*,(1-m-\rho-\sigma,h+k)\end{matrix}\right] \tag{1.12}$$

Provided that

$$\lambda_m=\frac{\Gamma(\lambda+m)}{\Gamma(\lambda)},\mathrm{Re}\left[\rho+h\left(\frac{b_j}{\beta_j}\right)\right].>-1\ ,\mathrm{Re}\left[\sigma+k\left(\frac{b_j}{\beta_j}\right)\right]>-1,h>0,k>0,\Omega>0,$$

$$|\arg z|<\frac{1}{2}\pi\Omega\quad ,\ \alpha>-1\ and\ \beta>-1\ ;\ where \tag{1.13}$$

$$\Omega\equiv\sum_{j=1}^{m}\beta_j+\sum_{j=1}^{n}\alpha_j-\sum_{j=m+1}^{q_i}\beta_{ji}+\sum_{j=n+1}^{p_i}\alpha_{ji}>0$$

The results (1.12) can be established by making use the well known Eulerian integral

$$\int_a^b (x-a)^{\rho-1}(b-x)^{\sigma-1}dx=(b-a)^{\rho+\sigma-1}B(\rho,\sigma) \tag{1.14}$$

$$\mathrm{Re}(\rho)>0\ ,\ \mathrm{Re}(\sigma)>0\ ,a,b\in R$$

The orthogonality property of the generalized prolate spheroidal wave function in the (slightly modified) form ([Gupta 1985], Eq.(3.1)) is

$$\int_a^b (x-a)^\alpha(b-x)^\beta\phi_n^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right]\phi_t^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right]dx=N_{n,t}^{\alpha,\beta}\delta_{n,t} \tag{1.15}$$

where

$$N_{n,t}^{\alpha,\beta}=\sum_{p=0}^{\infty}\left[R_{p,n}^{\alpha,\beta}(s)\right]^2(b-a)^{\alpha+\beta+1}\frac{\Gamma(n+p+\alpha+1)\Gamma(n+p+\beta+1)}{(2n+2p+\alpha+\beta+1)\Gamma(n+p+\alpha+1)} \tag{1.16}$$

And $\delta_{n,t}$ is the kronecker delta.

We also use the following lemma ([Gupta 1985], Eq.(2)) in our investigation:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k) \tag{1.17}$$

MAIN INTEGRAL

The following integral has been evaluated in this paper:

$$\int_a^b (x-a)^{\alpha}(b-x)^{\beta}\,\phi_n^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right]{}_RF_K\left[{}^{(e_R)}_{(f_K)};y(x-a)^{g}(b-x)^{\omega}\right]S_n^m\left[y_1(x-a)^{g_1}(b-x)^{\omega_1}\right]$$
$$I\left[z(x-a)^{h}(b-x)^{k}\right]dx=\sum_{p,q=0}^{\infty}(b-a)^{\rho+s+1+gq+\omega q}\,L(y)\,R_{\rho,n}^{\alpha,\beta}(s)\frac{(e_R)_q}{(f_K)_q}y^q$$
$$\sum_{m=0}^{n+p}\frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!\,(\alpha+1)_m}\quad I^*\left[z(b-a)^{h+k}\right] \tag{2.1}$$

where

$$I^*\left[z(b-a)^{h+k)}\right]=I_{p_i+2,q_i+1:r}^{m,n+2}\left[z(b-a)^{h+k}\middle|\begin{matrix}(-m-\rho-gq-g_1\alpha,h),(-\sigma-\omega q-\omega_1\alpha,k),A^*\\ B^*,(1-m-\rho-\sigma-gq-\omega q-(g_1+\omega)\alpha,h+k)\end{matrix}\right] \tag{2.2}$$

and

$$L(y)=\sum_{\alpha=0}^{[n/m]}\frac{(-n)_{m\alpha}}{\alpha!}A_{n.\alpha}\;y^{\alpha(b-a)(g_1+\omega_1)\alpha} \tag{2.3}$$

Provided that

$a\neq b,h,k>0,g>0,\omega>0,g_1>0,\omega_1>0,m$ is an arbitrary positive integer and the coefficients $A_{n,k}$ are arbitrary constant, real or complex,

$$\operatorname{Re}\left[\rho+h\left(b_j/\beta_j\right)\right]>-1,\operatorname{Re}\left[\sigma+k\left(b_j/\beta_j\right)\right]>-1\,,\Omega>0,\left|\arg z\right|<\frac{1}{2}\Omega\pi\,,\alpha>-1,$$

$\beta>-1\ m=n(or\,m>n+1,and\left|y\right|<1).$

PROOF

To establish (2.1), we express the generalized prolate spheroidal wave function as given in (1.2), the generalized hypergeometric function as infinite series ([Rainville 1960], Eq.(1.2)), and the general class of polynomials with the help of equation (1.9), change the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and sums involved in the process), then evaluated the inner integral by using (1.12) and interpreting the result by using (1.10), we arrive at the right hand side of (2.1).

SOLUTION OF THE PROBLEM

The solution of the problem stated in (1.5) is given by

$$u(x,t)=\sum_{p,q=0}^{\infty}\sum_{l,m=0}^{p}(b-a)^{\rho+\sigma 1+gq+\omega q}\,L(y)\,R_{p-l,l}^{\alpha,\beta}(s)\frac{(-p)_m(\alpha+\beta+p+1)_m}{m!(\alpha+1)_m}\left[N_{l,l}^{\alpha,\beta}\right]^{-1}$$

$$\frac{(e_R)_q}{(f_K)_q}\frac{y^q}{q!}I^*\left[z(b-a)^{h+k}\right]e^{-\left(p/q\right)(p+\alpha+\beta+1)l}\;\phi_l^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right]\qquad(3.1)$$

valid under the conditions mentioned with (2.1) and I^* is given by (2.2).

PROOF

The solution of (1.5) can be written as ([Gupta 1986], Eq.(2.3))

$$u(x,t)=\sum_{l=0}^{\infty}A_l\,e^{-B_l t}\,\phi_l^{\alpha,\beta}\left[s,\,1-2\frac{(x-a)}{(b-a)}\right]\qquad(3.2)$$

where $\phi_l^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right]$ is the general prolate spheroidal wave function given by (1.2), if t=0 then by the virtue of (1.6), we can obtain

$$B_l=\frac{(l+p)_m(l+p+\alpha+\beta+1)_m}{q}\ \ and\ \ A_l=\frac{G_l}{N_{l,l}^{\alpha,\beta}}$$

where

$$G_l=\int_a^b(x-a)^{\alpha}\,(b-x)^{\beta}\,\phi_l^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right]F(x)\,dx\qquad(3.3)$$

This is because the general prolate spheroidal wave function processes the orthogonality property (1.15).

Hence the solution of the problem (1.5) may be expressed as

$$u(x,t)=\sum_{l=0}^{\infty}G_l\left[N_{l,l}^{\alpha,\beta}\right]^{-1}\exp\left[-\frac{(l+p)(l+p+\alpha+\beta+1)}{q}\right]\phi_l^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right](3.4)$$

Now, if we take F(x) as given in (1.7) then by (3.2), we have formally

$$(x-a)^{\rho-\alpha}(b-x)^{\sigma-\beta}S_n^m\left[y_1(x-a)^{g_1}(b-x)^{\omega_1}\right]{}_RF_K\left[{(e_R)\atop (f_K)};y(x-a)^g(b-x)^{\omega}\right]$$

$$I^*\left[z(x-a)^h(b-x)^k\right]=\sum_{l=0}^{\infty}A_l\phi_l^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right]\qquad(3.5)$$

Equation (3.5) is valid since F(x) is continuous and of bounded variation in the open interval (a,b). Now multiplying both sides of (3.5) by

$$(x-a)^{\alpha}(b-x)^{\beta}\,\phi_l^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right],\alpha>-1\ \beta>-1$$

and integrate with respect to x from a to b, changing the order of integration and summation (which is permissible) on the right, we obtain

$$\int_a^b (x-a)^\rho (b-x)^\sigma \phi_l^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right] S_n^m\left[y_1(x-a)^{g_1}(b-x)^{\omega_1}\right] {}_RF_K\left[{}^{(e_R)}_{(f_K)}; y(x-a)^g(b-x)^\omega\right]dx$$

$$=\sum_{l=0}^{\infty} A_l \int_a^b (x-a)^\alpha (b-x)^\beta \phi_n^{\alpha,\beta}\left[s,1-2\frac{(x-a)}{(b-a)}\right]dx \tag{3.6}$$

Using orthogonality property of the generalized prolate spheroidal wave function (1.15) on right hand side and the result (2.1) on left hand side of (3.6), we obtain

$$A_l = \sum_{p,q=0}^{\infty} (b-a)^{\rho+\sigma+gq+\omega q} L(y) R_{\rho,n}^{\alpha,\beta}(s) \frac{(e_R)_q}{(f_K)_q} \frac{y^q}{q!} \sum_{m=0}^{n+p} \frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!(\alpha+1)_m}$$

$$\left[N_{l,l}^{\alpha,\beta}\right]^{-1} I^*\left[z(b-a)^{h+k}\right] \tag{3.7}$$

where I^* is defined by (2.2). On substituting the value of A_l from (3.7) in (3.5) and using (1.17), we arrive at the solution of the problem (1,5) as given in (3.1). If we take s=0 in (3.4), we get the solution of the problem in terms of the Jocabi polynomials. Again, setting $\alpha=\beta=0$ and s=0 in (3.4), we get the solution of the problem in terms of the Lengdre functions.

APPLICATION

We shall define the Riemann- Liouville derivative of function f(x) of order σ (or alternatively σ^{th} order fractional integral) [Srivastava *et al*. 1982] by

$$_aD_x^{-\sigma}\{f(x)\} = \begin{cases} \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_a^x (x-t)^{-\sigma-1} f(t)\,dt & , Re(\sigma)<0 \\ \dfrac{d^q}{dx^q}\, {}_aD_x^{\sigma-q}\{f(x)\} & , q-1 \le Re(\sigma) < q \end{cases} \tag{4.1}$$

where q is a positive integer and the integral exists.

For simplicity the special case of the fractional derivative operator $_aD_x^\sigma$ when a=0 will be written as D_x^σ. Thus we have

$$D_x^\sigma = {}_0D_x^\sigma \tag{4.2}$$

Now by setting b=x in (2.1), it can be rewritten as the following fractional integral formula

$$_aD_x^{-\beta-1}\left\{(t-a)^\alpha \phi_n^{\alpha,\beta}\left[s,1-2\frac{(t-a)}{(x-a)}\right] {}_RF_K\left[{}^{(e_R)}_{(f_K)}; y(b-a)^g(x-t)^\omega\right] S_n^m\left[y_1(t-a)^{g_1}(x-t)^{\omega_1}\right]\right.$$

$$\left. I\left[z(t-a)^h(x-t)^k\right]\right\}dx = \frac{1}{\Gamma(\beta+1)} \sum_{p,q=0}^{\infty} (x-a)^{\rho+s+1+gq+\omega q} L(y) R_{\rho,n}^{\alpha,\beta}(s) \frac{(e_R)}{(f_K)} \frac{y^q}{q!}$$

$$\sum_{m=0}^{n+p} \frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!\,(\alpha+1)_m} I^*\left[z(b-a)^{h+k}\right] \tag{4.3}$$

where $\mathrm{Re}(\beta+1)>0$ and all the condition of validity mentioned with (2.1) are satisfied.

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