▼ Journal of Research (Science), Bahauddin Zakariya University, Multan, Pakistan.
Vol. 18, No. 3, July 2007, pp. 167-176 ISSN 1021-1012

CONSTRUCTION OF BLOCK DESIGNS WITH NESTED ROWS AND COLUMNS

L. Rasul and I. Iqbal
Department of Statistics, B.Z. University, Multan, Pakistan.

Abstract
In row-column design the experimental units are grouped in two directions. These blocking may represented a further blocking factor or provide replications of the basic experimental designs. Incomplete block designs with nested rows and columns are designs for v treatments in s block of size k = pq < v. We describe a general method of constructing such designs using a set of cyclic shifts. This method is used to construct efficient designs. The majority of the designs given is new and extended previously published designs.

Keywords: Cyclic shift, fractional designs, incomplete block designs, lattice square nested row-column designs.

INTRODUCTION
Block designs with nested rows and columns were introduced by Singh and Dev [1979] as a generalization of Lattice square designs. These designs are incomplete block designs with v treatments and s blocks, each block containing pq treatments, each block being further grouped into p rows and q columns. It is convenient to call an arrangement of v treatments in s nonempty sets each of p rows and q columns (pq < v) a balanced incomplete block design with nested rows and columns if the following conditions are satisfied.

(i) every treatment, occurs at most once in a set.

(ii) given a pair of treatments

$$(p-1)\,\lambda_{r(i,j)} + (q-1)\,\lambda_{c(i,j)} - \lambda_{e(i,j)} = \lambda$$

where $\lambda_{r(i,j)}$, $\lambda_{c(i,j)}$ and $\lambda_{e(i,j)}$ denote the number of sets in which i and j occur in the same row, same column and same elsewhere respectively and λ is a constant independent of the pair of treatments chosen. In such a design every treatment occurs in exactly r sets where

$$r = \lambda(v-1)/\{(p-1)(q-1)\} \quad \text{clearly } vr = spq.$$

Singh and Dev [1979] also gave the following theorem.

THEOREM
For a equireplicate binary variance balanced design

$$pN_1N_1' + qN_2N_2' - NN' = \{r(p+q-1) - \lambda\}I + \lambda JJ'$$

where $\lambda(v-1) = r(p-1)(q-1)$

N = (r_{ij}) be the v × s matrix of elements r_{ij}, where r_{ij} denotes the number of times the ith treatment appears in the jth set (i = 1,2….v; j = 1,2….s). We may write N as N = (r_1,.....r_s) where r_j = (r_{1j},.....,r_{vj})' (j = 1,....,s). N_{ij} be the v×p incidence matrix of treatments versus rows in the jth set and N_{2j} be the v×p incidence matrix of treatments versus columns in the jth set (j=1…s). We can define N_1 = (N_{11}.....N_{1s})

N_2 = (N_{21}.....N_{2s})

Singh and Dev gave two methods of construction of such designs. Agrawal *et al.* [1982, 1983] gave different number of construction of these designs using the method of differences.

In this paper we gave the methods of construction of such designs using cyclic shifts. The method of cyclic shifts is defined below. For detail see Iqbal [1991].

METHOD OF CYCLIC SHIFTS

The v treatments are labeled as 0, 1, 2,...,v-1, and we consider the construction of equireplicate binary design for v treatments in b = v blocks of size k. The method of construction is to allocate to the first plot in the i-th block of the treatment i = 1, 2,..., v. We denote this using the vector u_1 = [0, 1,..., v-1]' which holds the treatments allocated to the first plot in each blocks 1, 2,..., v, respectively. To obtain the treatment allocation of the remaining plots in each block, we cyclically shift the treatments allocated to the first plot. In order to define a cyclic shift let u_j denote the allocation of treatment to the i-th plot in each block. That is, the i-th element of the u_j is the treatment allocated to plot i of block j. A cyclic shift of size q_j, when applied to plot i-1, is such that u_j = [u_{j-1} + q_j 1]' where addition is mod v, 1 is a vector of ones, 2 ≤ I ≤ k and 1 ≤ q_i ≤ v-1. Assuming that we always start with u_1 as defined above, a design is completely defined by a set of k-1 shifts, Q say, where Q = [q_1, q_2,...,q_{k-1}]. To avoid a treatment occurring more than once in a block we must ensure that the sum of any two successive shifts, the sum of any three successive shifts, …, the sum of any k-1 successive shifts is not equal to zero mod v. Subject to this constraint 0 may consist of any combination of shifts including repeats. Also the shift need only range from 1 to [v/2] inclusive, where [v/2] is the greatest integer less than or equal to v/2. This is because a shift of size q is equivalent to one of size [v-q] mod v.

To illustrate the above method of construction let us consider the construction of design for v = 5 and k = 3. The possible shifts are defined by Q = [q_1, q_2], where q_i = 1 or 2 and i = 1, 2. Two possible choices are Q_1 = [1,1] and Q_2 = [1,2]. Using Q_1 we get U_2 [1,2,3,4,0]' and U_3 = [2,3,4,0,1]'. The complete design obtained from using Q_1 is given below as Design 2.1 and the complete design obtained from using Q_2 is given below as Design 2.2. The blocks of design are written vertically.

Design 2.1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

Design 2.2

0	1	2	3	4
1	2	3	4	0
3	4	0	1	2

The properties of a design depend on the number of concurrence between the pairs of treatment. A concurrence between two treatments occurs when both treatments are in the same block. In Design 2.1 for example, treatments o makes the following numbers of concurrences with each of the treatments 1, 2, 3 and 4 respectively: 2, 1, 1, 2. In Design 2.1 the corresponding numbers are 1, 2, 2, 1. We will not continue with our comparison of these designs here but will do so below. Before doing so we note that

1. because of the cyclic nature of the construction, the number of concurrences between any treatment and the remainder can be obtained from the number of concurrences between treatment 0 and the remainder
2. as we have already mentioned, the number of concurrences between treatment 0 and the remainder can easily be obtained *from* Q, the set of shifts used to construct the design.

As an illustration of (1), we note that in Design 2.1 the number of concurrences between treatment 1 and treatments 2,3 and 4 are 2,1,1 respectively. That is, they are obtained by cycling the list of concurrences. To show that (2) is true, consider the number of concurrences for Design 2.1 and the shifts Q_1 = [1,1], used to construct the design. We first note that the number of concurrences are "symmetric" about [v/2] in the sense that any shift to size q that results in a concurrence between treatment 0 and treatment q also results in a concurrence between treatment 0 and v-q. This incidentally means that we need only use shifts that are such that $1 \leq qi \leq [v/2]$ when defining Q.

Secondly, if shifts q_1 and q_2, for example, are applied successively to treatment 0, the result is a concurrence between treatment 0 and treatment q_1 and q_2 and a concurrence between treatment 0 and treatment $q_1 + q_2$. If a third shift, q_3 say, is then applied after q_1 and q_2, the following treatments will also concur with treatment 0: q_3, $q_2 + q_3$ and $q_1 + q_2 + q_3$. This adding of shifts to get the treatments which concur with 0 works for the general case and so enables the number of concurrences of a design to be obtained directly from the shifts which define it. This is a very useful result which enables the properties of a design to be determined very quickly and easily and avoid to having to construct the blocks of the design. In general case, if shifts q_1, q_2,... ,q_{i-1} have been applied successively to treatment 0, then the additional concurrences which result when shift q_i is applied are between treatment 0 and treatment $q_1 + q_2 ... q_j$, $q_2 + q_3 + q_j$ $q_{i-1} + q_i$, q_i, where addition is mod v.

To illustrate the calculation of the numbers of concurrences from the shifts, let us consider Design 2.1 and 2.2 as given earlier. The shifts used to construct Design 2.1 are Q_1 = [1,1]. The treatment which concur with treatment 0 are therefore 1,1 and 1 +1, i.e., treatment 1 concurs twice and treatment 2 concurs once. By the symmetry property, treatment 0 also concurs twice with treatment 4 and once with treatment 3. The concurrences of treatment 1 with treatments 2, 3 and 4 are 2, 1, 1 respectively. Similarly, the concurrences of treatment 2 with treatments 3 and 4 are 2 and 1 respectively, and so on. In Design 2.2 the set of shift is Q_2 = [1,2] and so the treatments which concur with treatment 0 in this design are 1, 2 and 1 + 2. By the symmetry property a concurrences with 3 also implies a concurrence with 2 and so treatment a concurs the following number of times with treatment 1, 2, 3 and 4, respectively: 1, 2, 2, 1. We can also find the concurrences of other treatment pairs as we did for Design 2.1. For more detail about this method see Iqbal [1991].

Some choice of shifts lead to what we refer to as fractional designs. Such designs are described in the following section.

FRACTIONAL DESIGNS

A feature of the method of construction is that certain sets of shifts produce designs that are made up of complete replicates of smaller designs. That is the v blocks can be divided into s sets of size n = vis and the n blocks in each set contain the same treatment allocation. As for example, consider v = 6, k = 4 and Q = [1, 2, 1]. The design obtained by applying Q to U1 is given as Design 2.1.1 below:

Design 2.1.1

0	1	2	3	4	5
1	2	3	4	5	0
3	4	5	0	1	2
4	5	0	1	2	3

It can be seen that Design 2.1.1 is made up of two complete designs as indicated by the vertical line. The number of concurrences that treatment a makes with each of the other treatments are respectively 2, 2, 4, 2, 2 in the whole design, whereas the concurrences are 1.1.2.1.1 (half) in the fractional design.

In order to decide if a set of shifts will produce a fractional design we take the value of v/k and determine the smallest integer z say, which makes (v x z)/k an integer n say. If v is also divisible by n then m fractional designs can be obtained. Let v/n = m, then we have m fractional designs within the whole design with z the number of replicates of each fractional design and n the number of blocks of each fractional design. The shifts that will produce the design we require must be such that z successive shifts add to n. In the above example we have

$$v/k = 6/4$$
$$(v \times z)/k = (6 \times 2)/4 = 3 = n$$
$$v/n = 6/3 = 2 = m$$

i.e. m = 2 and there are two fractional designs within the whole design. Each fractional design has n (= 3) blocks, z (= 2) replications and the shifts used are Q = [1, 2, 1] i.e. the sum of z (= 2) successive shifts is equal to n (= 3).

So for, we have only considered the construction of Designs for b ≤ v. To construct designs for b > v, we can combine two or more full or fractional designs to get a new design.

ADDING A CONSTANT TREATMENT

Some times a design for v treatments with block of size k and k -1 can be converted into design for v treatments and all blocks of size k by adding additional treatment to each of the smaller blocks of size k -1, Design 2.2.1, below, is an example of such a design for v = 8, b=14 and A = 3. The two sets of shifts used to construct the original design for v = 7, k_1 = 4 and k_2 = 3 were Q_1 = [1,2,3] and Q_2 = [2,4]. We note that treatment 7 has been added to each block of the smaller design.

Design 2.2.1

0	1	2	3	4	5	6	0	1	2	3	4	5	6
1	2	3	4	5	6	0	2	3	4	5	6	0	1
3	4	5	6	0	1	2	6	0	1	2	3	4	5
6	0	1	2	3	4	5	7	7	7	7	7	7	7

METHOD OF CONSTRUCTION

V = 5, P = q = 2, b = 1

To construct the design for above parameters we need one set of shifts. Each set consists of 3 shifts. The possible sets of shifts are given below:

(1,1,1), (1,1,2), (1,2,1), (1,2,4), (1,3,3), (1,3,4), (2,1,1), (2,1,3), (2,2,2), (2,2,4), (2,4,2), (2,4,3), (3,1,2), (3,1,3), (3,3,1), (3,3,3), (3,4,2), (3,4,4,), (4,2,1), (4,2,2), (4,3,1), (4,3,4), (4,4,3), (4,4,4).

If we consider a incomplete block design for V = 5 and K = 4 the resultant design is balanced, when we construct it by using any set of shifts.

Now we want a design which is also nested row and column. The condition for row wise balanced is that in the new set, consists of first and third shift (also its complement) each shift must appear equal number of times. A design is column wise balanced if in the new set consists of the sum of *two* successive shifts mod v (also its complement) each shift appears equal number of time. If we examine above all possible shifts we note that it is not possible to have one set of shifts, which give a design, balanced both row wise and column wise. e.g. Set of shift [1,3,3] give row wise balanced design but not column wise. The design is given below:

Design 3.1

0	1	1	2	2	3	3	4	4	0
4	2	0	3	1	4	2	0	3	1

The set of shift [1,2,4] give a column wise balanced design but not row wise. The design is given below:

Design 3.2

0	1	1	2	2	3	3	4	4	0
3	2	4	3	0	4	1	0	2	1

The set of shifts [1,2,1] give a design which is neither row wise balanced nor column wise balanced but balanced nested row column design. The design is given below.

Design 3.3

0	1	1	2	2	3	3	4	4	0
3	4	4	0	0	1	1	2	2	3

If we combine two designs given below, we obtain a design, which is balanced row wise as well as column wise, i.e. we have a design for V = 5, K = 4, P = q = 2, b = 2 Sets of shifts = [1,3,3] + [2,1,1] the following set of parameters.

Design 3.4

0	1	1	2	2	3	3	4	4	0
4	2	0	3	1	4	2	0	3	1
0	2	1	3	2	4	3	0	4	1
3	4	4	0	0	1	1	2	2	3

Now we give only the set (S) of shifts used to construct the designs for different values of v, p, q and r. For complete designs see Rasul [2002].

V = 6 K = 4, P = q = 2

r	Set of shifts
2	(1,2,1)1/2
4	(1,1,2)
6	(1,1,1)+(1,2,1)1/2
8	(1,1,2) + (1,2,1)1/2 + (1,3,S)1/2

V = 7

r	Set of shifts
4	(1,2,3)
8	(1,2,3)+(1,1,2)
12	(1,1,1)+(2,2,2)+(3,3,3)

V = 8

r	Set of shifts
2	(4,5,4)1/2
4	(1,4,2)
6	(1,4,2) + (4,S,4)1/2
8	(2,1,1) + (1,2)1/2

V = 9

r	Set of shifts
4	(1,2,2)
8	(1,2,2) + (3,7,4)

V = 10

r	Set of shifts
2	(1,S,9)1/2
4	(1,2,6)
6	(1,2,6) + (1,4,1)1/2
8	(1,2,6) + (2,3,4)
10	(1,2,6) + (1,5,2) + (1,4,1)1/2
12	(1,2,6) + (1,5,2) + (2,3,4)
14	(1,2,6) + (1,5,2) + (2,3,4) + (1,4,1)1/2
16	(1,2,6) + (1,5,2) + (1,3,5) + (2,3,4)
18	(1,2,6) + (1,2,6) + (1,5,2) + (2,3,4) + (1,4,1)1/2

V = 11

r	Set of shifts
4	(1,3,5)
8	(1,3,5) + (2,4,10)
12	(1,3,5) + (2,4,10) + (4,4,2)
16	(1,2,2) + (1,3,5) + (2,4,10) + (4,4,2)
20	(1,2,2) + (1,3,2) + (1,3,5) + (2,4,10) + (4,4,2)

V = 12

r	Set of shifts
4	(1,2,6)
8	(1,2,6) + (2,3,8)
12	(1,2,2) + (1,2,6) + (2,3,8)

V = 13

r	Set of shifts
4	(2,1,4)
8	(2,1,4) + (3,11,8)
12	(2,1,4)+(3,11,8)+(6,5,12)

V = 14

r	Set of shifts
2	(1,7,13)1/2
4	(1,2,2)
6	(2,7,4)+(1,7,13)1/2
8	(1,2,2) + (4,2,5)
10	(1,2,2) + (4,2,5) + (1,7,13)1/2
12	(1,2,2)+(2,2,3)+(5,1,7)
14	(1,2,2) + (2,2,3) + (4,2,5) + (1,7,13)1/2
16	(1,1,5) + (1,2,2) + (2,2,3) + (5,1,7)
18	(1,1,5) + (1,2,2) + (2,2,3) + (5,1,7) + (7,6,7)1/2
20	(1,1,5) + (1,2,2) + (2,1,3) + (4,2,5) + (5,1,1)

22	(1,1,5) + (1,2,2) + (2,1,3) + (4,2,5) + (5,1,1) +(7,6,7)1/2
24	(1,1,5) + (1,2,2) + (2,1,3) + (3,1,5) + (4,2,5) + (5,1,1)
26	(1,1,5) + (1,2,2) + (2,1,3) + (3,1,5) + (4,2,5) + (5,1,1) +(7,6,7)1/2

V = 15

r	Set of shifts
4	(1,2,2)
8	(1,2,2) + (5,2,4)
12	(1,1,5) + (1,2,2) + (5,2,4)
16	(1,1,5) + (1,2,2) + (5,2,4) + (12,1,8)
20	(1,1,5) + (1,2,2) + (1,2,5) + (5,2,4) + (12,1,8)
24	(1,1,5) + (1,2,2) + (1,2,5) + (5,2,4) + (8,2,9) + (12,1,8)
28	(1,1,5) + (1,2,2) + (1,2,5) + (1,2,5) + (5,2,4) + (8,2,9) + (12,1,8)
32	(1,1,5) + (1,2,2) + (1,2,5) + (1,2,5) + (2,2,4)+{5,2,4)+(8,2,9)+(12,1,8)

V = 16

r	Set of shifts
1	(4,4,4)1/4
2	(1,8,15)1/2
3	(1,8,15)1/2 + (4,4,4)1/4
4	(1,2,2)
5	(1,2,5) + (4,4,4)1/4
6	(2,3,10)+(1,8,15)1/2
7	(2,3,3) + (1,8,15)1/2 + (4,4,4)1/4
8	(1,2,2)+(5,1,7)
9	(1,2,2) + (5,1,6) + (4,4,4)1/4
10	(1,2,2)+(5,1,7)+(1,8,15)1/2
11	(1,2,2) + (5,1,6) + (1,8,15)1/2 + (4,4,4)1/4
12	(1,2,2) + (1,2,2) (5,1,7)
13	(1,2,2) + (1,4,3) + (5,1,6) + (4,4,4)1/4
14	(1,2,2) + (2,3,3) + (5,1,7) + (1,8,15)1/2
15	(1,2,2) + (1,2,2) + (5,1,7) + (5,1)t

V = 17

r	Sets of shifts
4	(1,2,2)
8	(1,2,2) + (5,2,6) .

V = 18

r	Sets of shifts
2	(1,9,17)1/2
4	(1,2,2)
6	(1,2,2) + (1 ,9,17)1/2
8	(1,2,2) + (5,1,7)
10	(1,2,2) + (5,1,7) + (9,14,9)1/2
12	(2,2,1) + (7,1,5) + (9,1,2)
14	(2,2,1) + (7,1,5) + (9,1,2) + (1,9,17)1/2
16	(2,2,1) + (7,1,5) + (9,1,2) + (4,1,6)
18	(2,2,1) + (7,1,5) + (9,1,2) + (4,1,6) + (1,9,17)1/2
20	(1,2,2) + (2,2,1) + (4,1,6) + (7,1,5) + (9,1,2)
22	(1,2,2) + (2,2,1) + (4,1,6) + (7,1,5) + (9,1,2) + (1,9,17)1/2
24	(1,2,2) + (2,2,1) + (4,1,6) + (7,1,5) + (7,1,5) + (9,1,2)
26	(1,1,3) + (1,2,2) + (2,2,1) + (5,1,7) + (6,2,7) + (7,1,5) + (9,1,2)1/2
28	(1,2,2) + (2,2,1) + (4,1,6) + (7,1,5) + (7,1,5) + (9,1,2) + (1,9,17)1/2

	+ (9,14,9)1/2
30	(1,1,3) + (1,2,2)+(2,2,1)+(2,2,3)+(5,1,7)+(6,2,7)+(7,1,5)+(9,14,9)1/2
32	(1,1,3) + (1,2,2) + (2,2,1) + (2,2,3) + (5,1,7) + (7,1,5) + (6,2,7) + (1,9,17)1/2 + (9,14,9)1/2
34	(1,1,3) + (1,2,2) + (1,2,2) + (2,2,1) + (5,1,7) + (5,1,7) + (7,1,5) + (6,2,7) + (9,14,9)1/2

Designs for p = 2, q = 3

For the above case the new set of shifts can be find as follows. If in the new set each shift appears an equal number of times then the design will be balanced nested rows columns.

For row wise balance the new set of shifts is S_{i1}, S_{i2}, $S_{i1} + S_{i2}$, S_{i4}, S_{i5}, $S_{i4} + S_{i5}$ (also symmetric shifts) where i is the number of sets of shifts. If we construct the design by adding a constant treatment then the new set of shifts will be S_{i1}, S_{i2}, $S_{i1} + S_{i2}$, S_{i4}.

Similarly for column wise balance the new set of shifts is $S_{i1} + S_{i2} + S_{i3}$, $S_{i2} + S_{i3} + S_{i4}$, $S_{i3} + S_{i4} + S_{i5}$ (also symmetric shifts).

If we construct the design by adding a constant treatment the new set of shifts will be $S_{i1} + S_{i2} + S_{i3}$, $S_{i2} + S_{i3} + S_{i4}$ (also symmetric shifts).

V = 8, P = 2, q = 3

r	Set of shifts
3	(1,2,4,6,7)1/2
6	(1,1,2,2,3)
9	(1,4,5,2,2) + (2,3,4,5,6)1/2
12	(1,4,5,2,2) + (1,2,4,6, 7)1/2 + (2,3,4,5,6)1/2
15	(1,2,3,1,3) + (1,4,5,2,2) + (2,3,4,5,6)1/2
18	(1,1,2,2,3) + (1,2,3,1,3) + (1,4,5,2,2)
21	(1,1,2,2,3) + (1,2,3,1,3) + (1,4,5,2,2) + (2,3,4,5,6)1/2
24	(1,1,2,2,3)+(1,4,5,2,2) + (3,3,1,2,1) + (1,1,2,1,1)1/2 + (2,3,4,5,6)1/2

Designs for p = 3, q = 2

For the above case the new set of shifts are find as follows.

For column wise balance the new set of shifts is $S_{i1} + S_{i2}$, $S_{i2} + S_{i3}$, $S_{i3} + S_{i4}$, $S_{i4} + S_{i5}$, $S_{i1} + S_{i2}$, $S_{i3} + S_{i4}$, $S_{i2} + S_{i3} + S_{i4} + S_{i5}$, (also symmetric shifts). For row wise balance the new set of shifts is S_{i1}, S_{i3}, S_{i5} (also symmetric shifts) when we construct such design, by adding a constant treatment, we do not take S_{i5}.

V = 8, P = 3, q = 2

r	Set of shifts
3	(1,2,4,6,7)1/2
6	(1,1,1,1,2)
9	(1,1,1,1,2) + {1,2,4,6,7)1/2
12	(1,1,1,1,1) + (1,2,2,7,4)
15	(1,1,1,1,1) + (1,1,2,7,4) + {1,2,4,6,7)1/2
18	(1,1,1,1,1) + (1,2,3,4,4) + (1,2,3,4,4)
21	(1,3,2,6,4) + {1,1,1,1)1/2 + {2,2,2,2)1/2 + {3,3,3,3)1/2

Designs for p = 3, q = 3

For the above case, a design is balance column wise if the new set of shifts (define below) contains each shift an equal number of time. $S_{i1} + S_{i2} + S_{i3}$, S_{i2} +

$S_{i3} + S_{i4}$, $S_{i3} + S_{i4} + S_{i5}$, $S_{i4} + S_{i5} + S_{i6}$, $S_{i5} + S_{i6} + S_{i7}$, $S_{i6} + S_{i7} + S_{i8}$, $S_{i1} + S_{i2} + S_{i3} + S_{i4} + S_{i5} + S_{i6}$, $S_{i2} + S_{i3} + S_{i4} + S_{i5} + S_{i6} + S_{i7}$, $S_{i3} + S_{i4} + S_{i5}$, $S_{i6} + S_{i7} + S_{i8}$ (also symmetric values). For row wise balance the new set of shifts is S_{i1}, S_{i2}, $S_{i1} + S_{i2}$, S_{i4}, S_{i5}, $S_{i4} + S_{i5}$, S_{i7}, S_{i8}, $S_{i7} + S_{i8}$ (also symmetric values).

V = 9, P = 3, q = 3

r	Set of shifts
3	(1,1,1,2,3,5,2,1)1/3
6	(1,1,1,2,3,5,2,1)1/3 + {2,3,2,5,1,1,1,1)1/3
9	(1,2,8,2,3,8,8,3)
12	(1,2,8,2,3,8,8,3) + (2,3,2,5,1,1,1,1)1/3

CONCLUSION

In this paper we have given general conditions which the sets of shifts have to satisfy in order that they can be used to construct the designs. As well as showing that existing designs can be constructed by this method, we also give many new designs.

Singh and Dev [1979] give the following design for v = 5, r = 4, p = q = 2 and λ =1

1 2	1 5	1 4	1 3	2 4
3 4	2 3	5 2	4 5	3 5

This design is not variance balanced because

$pN_1N_1' + q\,N_2N_2' - NN' \neq \{r(p+q-1)-\lambda\}I + \lambda JJ'$

$$pN_1N_1' + q\,N_2N_2' - NN' =$$

$$2\begin{pmatrix}4&1&1&1&1\\1&4&1&1&1\\1&1&4&1&1\\1&1&1&4&1\\1&1&1&1&4\end{pmatrix}+2\begin{bmatrix}4&1&1&1&1\\1&4&1&2&0\\1&1&4&0&2\\1&2&0&4&1\\1&0&2&1&4\end{bmatrix}-\begin{bmatrix}4&3&3&3&3\\3&4&3&3&3\\3&3&4&3&3\\3&3&3&4&3\\3&3&3&3&4\end{bmatrix}$$

$$=\begin{bmatrix}12&1&1&1&1\\1&12&1&3&-2\\1&1&12&-2&3\\1&3&-2&12&1\\1&-2&3&1&12\end{bmatrix}$$

We can construct the above design by using the set *of* shifts [1,2,1]. The design is

0	1	1	2	2	3	3	4	4	0
3	4	4	0	0	1	1	2	2	3

This design is variance-balanced design because

$$PN_1N_1' + qN_2N_2' - NN' = [r(p+q-1]I + \lambda JJ'$$

$$where\,\lambda = \frac{r(p-1)(q-1)}{v-1} = \frac{4(2-1)(2+1)}{4} = 1\ \ \lambda = 1$$

$pN_1N_1' + q\,N_2N_2' - NN'$

$$= 2\begin{pmatrix} 4 & 2 & 0 & 0 & 2 \\ 2 & 4 & 2 & 0 & 0 \\ 0 & 2 & 4 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 2 & 0 & 0 & 2 & 4 \end{pmatrix} + 2\begin{pmatrix} 4 & 0 & 2 & 2 & 0 \\ 0 & 4 & 0 & 2 & 2 \\ 2 & 0 & 4 & 0 & 2 \\ 2 & 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 3 & 3 & 3 & 3 \\ 3 & 4 & 3 & 3 & 3 \\ 3 & 3 & 4 & 3 & 3 \\ 3 & 3 & 3 & 4 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 12 & 1 & 1 & 1 & 1 & 1 \\ 1 & 12 & 1 & 1 & 1 & 1 \\ 1 & 1 & 12 & 1 & 1 & 1 \\ 1 & 1 & 1 & 12 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 12 \end{pmatrix}$$

$$[r(p+p-1)-\lambda]I + \lambda JJ' = [4(2+2-1)-1]I + 1JJ'$$

$= 11I + 1JJ'$

$$= 11\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 1 & 1 & 1 & 1 \\ 1 & 12 & 1 & 1 & 1 \\ 1 & 1 & 12 & 1 & 1 \\ 1 & 1 & 1 & 12 & 1 \\ 1 & 1 & 1 & 1 & 12 \end{bmatrix}$$

Hence

$$PN_1N_1' + qN_2N_2' - NN' = [r(p+q-1)-\lambda]I + \lambda JJ'$$

References

Agrawal, H.L. and Prasad, J. (**1982**) “Some methods of constructions of balanced incomplete block designs with nested rows and columns”, *Biometrika* **69**, 481-483.

Agrawal, H.L. and Prasad, J. (**1983**) “On constructions of balanced, incomplete block designs with nested rows and columns”, *Sankhya* **B 30**, 345-350.

Iqbal, I. (**1991**) “Construction of Experimental Designs using Cyclic Shifts”, PhD. Thesis, University of Kent at Canterbury, Kent, UK.

Rasul, L. (**2002**) “Construction of block designs with nested rows and columns”, M. Phil. Thesis, B.Z.University, Multan, Pakistan.

Singh, M. and Dev (**1979**) “Block designs with nested rows and columns”, *Biometrika* **66**, 321-326.