

IMPACT OF SOLAR SPIN ON PLANETARY ORBITS

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Abstract

This paper examines the possible effect of solar spin in the planetary orbit. We show in our earlier paper that if we incorporate the contribution of spin of the central gravitating body in orbital calculations, a residual slight perturbation on the standard constant areal velocity should exist. In particular, the second law of planetary motion requires a revision. However, it turns out that the classical result of Kepler is recoverable from our result as a special case. To be able to appreciate the need for the revision suggested by the new perturbation considered here, this paper looks into the genesis of orbital theory. We herein propose to reduce spin theory to non-relativistic regime. In fact, we consider restricted three-body problem. Confining, for the moment, again to the specific context of solar system, our initial calculations show that the transverse component the transverse component of the force field is nonzero, in contrast to the GN-physics (Galilei-newtonian physics) wherein such a component vanishes. In particular, the transverse component of the central force field does vanish if we neglect the spin of the gravitating star. This situation is radically different from that of GN-theory (where linearisation often does result). However, if we set the spin equal to zero, we retrieve the orbit equation of GN-physics. As regards solution, we here apply numerical schemes to determine solution of nonlinear orbit equation for Earth. Our results exhibit that the new light on issue in relativistic celestial mechanics and models of planetary motion.

Keywords: Kepler's laws, orbits, solar spin.

INTRODUCTION

It is a truism that the tools invented to analyse astrodynamics have turned out to have astonishing applicability in diverse fields. Even today, there are significant problems asking for their solution. In particular, stellar spin — as established by the classical and modern observations related to dynamics of sunspots — in the context of general orbital theory is an important problem which has not yet been addressed by existing formalisms [Bray and Loughhead 1964, Meadow 1970]. As such, orbital / satellite dynamics around a spinning celestial body should in principle take into account the possible effect of spin of the central gravitating body [Brumberg 1985, Laskar 1999, 2000, 2004]. For instance, from the point of

view of enhanced precision attainable via the high-Q technology emerging now, relativistic correction of standard theories of celestial mechanics can no more be ignored [Anderson 1998].

Albeit the Schwarzschild solution of Einstein field equations is generally used to do celestial mechanics, it has limited scope and, moreover, it is not a representative solution for the purpose [Misner *et al.* 1970]. Arguably, one may use the Kerr black hole solution of Einstein field equations for the description of a normal star in relativistic celestial mechanics [Kerr 1963]. Clearly, an incorporation of the spin correction in orbital theory needs to be carried out in the framework of modern gravitation physics [Brumberg 1985]. We show in our earlier paper that if we incorporate the contribution of spin of the central gravitating body in orbital calculations, a residual slight perturbation on the standard constant areal velocity should exist [Iqbal and Quamar 2003]. In particular, the second law of planetary motion requires a revision.

To be able to appreciate the need for the revision suggested by the new perturbation considered here, we need to look into the genesis of orbital theory. This project attempts to construct the planetary orbit equation, which includes the possible effect of solar spin. As regards solution, we apply numerical schemes to determine solution of nonlinear orbit equation for Earth. The plan of the paper is as follows. In next section, review of earlier paper [Iqbal and Quamar 2003] is presented, which shows that the second law of planetary motion might require revision. In proceeding section, the planetary orbit equation has been worked out along with its numerical solution. Final section comprises a summary and conclusions.

MODIFICATION OF KEPLER'S SECOND LAW DUE TO THE SPIN OF CENTRAL GRAVITATING SOURCE

It is an empirical fact, based on the solar physics of sunspots, that the sun and generally stars are rotating gravitating sources for bodies of the respective attendant solar systems. The gravitational field of any such source may be represented by a class of rotating space-time manifolds. As is well known, from the viewpoint of symmetry structure of space-time manifolds, metrics of such a class of manifolds are axisymmetric in character. However, nautical almanacs and ephemerides currently prepared at various observatories around the globe do not take this effect into account. Thus a planetary theory for the gravitational field of a star (or a satellite dynamics around a spinning planet [Correia *et al.* 2003], including the problem of artificial satellites) should in principle take into account the possible effect of axial symmetry. To keep mathematics tractable, in the first instance, we considered the needed incorporation of spin correction in the class of axisymmetric space-time manifolds possessing staticity in an earlier paper [Iqbal and Quamar 2003]. To be able to interpret things physically, we carried out the analysis of orbital evolution equations by invoking the concept of relativistic multipole moments of the gravitating source as a perturbation on static axisymmetric space-times.

However, what is nearer the ground reality is the class of axisymmetric stationary fields studied in relativistic gravitodynamics (RGD) [Misner *et al.* 1970]. We employ the class of Kerr space-times [Kerr 1963] to incorporate the effects of the spin of the central body of any solar system into orbit calculations of a satellite or

planet. Kerr space-time metric tensor in Boyer and Lindquist coordinates [Boyer and Lindquist 1967] is

$$ds^2 = \left\{ 1 - 2M r \Lambda^{-1} \right\} dt^2 - 4aM r \sin^2 \theta \Lambda^{-1} dt d\phi \\ - \left\{ 2a^2 M r \sin^2 \theta \Lambda^{-1} + r^2 + a^2 \right\} \sin^2 \theta d\phi^2 \\ - \Lambda (\Lambda^{-1} dr^2 + d\theta^2), \quad (1)$$

where

$$\Lambda \equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2Mr + a^2, \quad (2)$$

In Eq. (1), we interpret M and Ma as mass and angular momentum. We obtain the following orbit equation employing Hamilton-Jacobi equations [Iqbal and Quamar 2003]

$$\Lambda^2 \frac{d\theta}{d\tau} = \left[C - \cos^2 \theta \left\{ a^2 (m - E^2) + \frac{L_z^2}{\sin^2 \theta} \right\} \right]^{1/2}, \quad (3)$$

$$\Lambda^2 \frac{dr}{d\tau} = \left(\left[E \{ r^2 + a^2 \} - L_z a \right]^2 \right. \\ \left. - \Delta \{ m^2 r^2 + (L_z - aE)^2 + C \} \right)^{1/2}, \quad (4)$$

$$\Lambda^2 \frac{d\phi}{d\tau} = - \left[aE - \frac{L_z}{\sin \theta} \right] \\ + \frac{r^2 + a^2}{\Lambda} \left[E (r^2 + a^2) L_z a \right], \quad (5)$$

$$\Lambda^2 \frac{dt}{d\tau} = -a (aE \sin^2 \theta - L_z) + \frac{a}{\Delta} \left[E \{ r^2 + a^2 \} L_z a \right]. \quad (6)$$

It is assumed that motion takes place in the equatorial plane ($\theta = \pi/2$). As regards Eq (5), it sheds new light regarding conclusions of Galilei-newtonian (GN) orbital theory. Let us look at possible consequences of the modification of classical orbital theory obtained. Notice that in the equatorial plane, Eq (5) assumes the following form:

$$r^2 \frac{d\phi}{d\tau} = r \frac{\{ r - 2M \} L_z + 2aME}{\Delta}. \quad (7)$$

This shows that areal velocity is not, in general, constant due to spinning gravitating sources as described by vacuum gravitational field solutions given by Eq. (1), again in contrast to the case of Schwarzschild class of space-times currently used for orbit calculations, for instance those required in the construction of ephemerides and nautical almanacs. By setting $a = 0$ in the preceding equation, there results

$$r^2 \frac{d\phi}{d\tau} = L_z. \quad (8)$$

In other words, areal velocity becomes constant on setting solar angular momentum equal to zero, a situation not corroborated by astrophysical observations and experimentation. The calculations, therefore, show that if we

incorporate the contribution of spin of the central gravitating body in orbital calculations, a residual slight perturbation on the standard constant areal velocity should exist. In particular, the second law of planetary motion requires a revision. However, it turns out that the classical result of Kepler is recoverable from the present result as a special case.

APPLICATIONS OF MODIFIED KEPLER'S SECOND LAW

To be able to appreciate the need for the revision suggested by the new perturbation considered here, we need to look into the genesis of orbital theory. As it is known that orbital mechanics of solar systems, stellar systems (like our own Milky-Way Galaxy), galaxy clusters, etc. is essentially based on celestial mechanics. But this whole edifice presently rests on the conventional (empirical) keplerian laws — based on Brahe's planetary observations — that describe motion in unperturbed planetary orbits [Lagrange 1772]. Moreover, when it refers to Keplerian orbits, we implicitly assume that masses of planets are truly negligible and that Kepler's so-called "laws" are exact. In fact, however, with the exception of two-body motion (an n -body problem for the specific subcase of $n = 2$), astrodynamical problems are, generally, incapable of exact analytical solutions [Battin 1985].

Due to the difficulty of absence of an exact solution to 3-body and, generally, n -body problem, one often tries to exploit the method of two-body problem — this is particularly true of applications of standard GN-theory. This same difficulty is perhaps also responsible for the popular misconception that planets of our solar system have constant areal velocity. In fact, however, staying right in the framework of GN-theory, if we switch from the two-body problem to even the restricted three-body problem, areal velocity turns out to be nonconstant in general. This situation nicely compares with our finding of the general nonconstancy of the areal velocity (MKSL) in relativistic astrodynamics.

Confining, for the moment, again to the specific context of solar system, note that Eq. (7) can be written as

$$\frac{d\varphi}{d\tau} = \frac{\{r - 2M\}L_z + 2aME}{r\Delta}. \quad (9)$$

Again, as we know, the radial and transverse components of the central force are as follows:

$$F_R = m \left[\frac{d^2 r}{d\tau^2} - r \left(\frac{d\varphi}{d\tau} \right)^2 \right], \quad (10)$$

$$F_T = \frac{m}{r} \frac{d}{d\tau} \left((r)^2 \frac{d\varphi}{d\tau} \right). \quad (11)$$

Sandwiching Eqs (7) and (9) together then yields

$$F_T \neq 0.$$

This shows that, in the new development presented in preceding section, the transverse component of the force field is nonzero, in contrast to the GN-physics wherein such a component vanishes. In particular, the transverse component of the central force field does vanish if we neglect the spin of the gravitating source.

It suggests that the inverse square law, as generally employed in GN-physics, needs a modification.

Yet another facet of MKSL emerges from the fact that the existence of nonzero transverse components of the force is mathematically equivalent to the existence of a third body in the physical system being considered. For mathematical tractability, let us treat here the 3-body problem as a restricted 3-body problem. When all of the masses are finite, the three-body problem admits of certain exact solutions in which the ratios of the mutual distances of the bodies are constant. However, if one of the masses is infinitesimal so that it has no appreciable effect on the motion of the other two, then the possible motions of the small mass are considerably expanded. This is the restricted problem of three bodies, examples of which are approximated by a spacecraft in the earth-moon system or a planetary satellite in the planet system. We assume that the non-zero transverse force is cancel out with the force of the central body, which exerts on infinitesimal body. For such a problem, the force F is given by

$$F = \frac{1}{r} (m M + m_0 M + m m_0), \quad (12)$$

where m , M and m_0 are masses of the Earth, Sun and infinitesimal body [Battin 1985]. For convenience, we suppose that three bodies are located at an equilateral triangle. Notice that the resultant orbit equation turns out to be

$$(m M + m_0 M + m m_0) = m r \left[\frac{d^2 r}{d\tau^2} - r \left(\frac{d\varphi}{d\tau} \right)^2 \right] \quad (13)$$

Plugging Eq. (9) into this equation yields the following second order nonlinear orbit equation:

$$(m M + m_0 M + m m_0) = m r \left[\frac{d^2 r}{d\tau^2} - \frac{\{(r - 2 M) L_z + 2 a M e\}^2}{r \Delta^2} \right] \quad (14)$$

Notice that this is a nonlinear orbit equation, with apparently no convenient transformation in sight which could linearise it. This situation is radically different from that of GN-theory (where linearisation often does result). However, if the spin is set equal to zero in Eq. (14), the orbit equation of GN-physics is retrieved. As regards, solution approaches to Eq. (14), its numerical solution using RK method [Gupta *et al.* 1985, Kintoshita and Nakai 1989] can be determined. The earth orbit has been plotted in Fig. 1 which shows that earth follows an elliptical orbit (closed orbit).

SUMMARY AND CONCLUSION

As shown in introduction, the issue of enhancing our ability to carry operations into the interplanetary environment for jobs like utilizing the material and energy resources of space and improving our ecosphere and biosphere, signals a paradigm shift in the current space science programmes. Clearly, this rests among other factors on the fundamental framework for launching artificial satellites and space probes in various kinds of orbits as envisaged by the current and future space science missions of various countries. Although Keplerian laws of GN-physics have had fundamental place so far, they are not altogether immutable and require a revision especially in the light of modern gravitation

physics and, in particular, the new perturbation, the spin, of the central gravitating body controlling the orbital motion of the attendant objects.

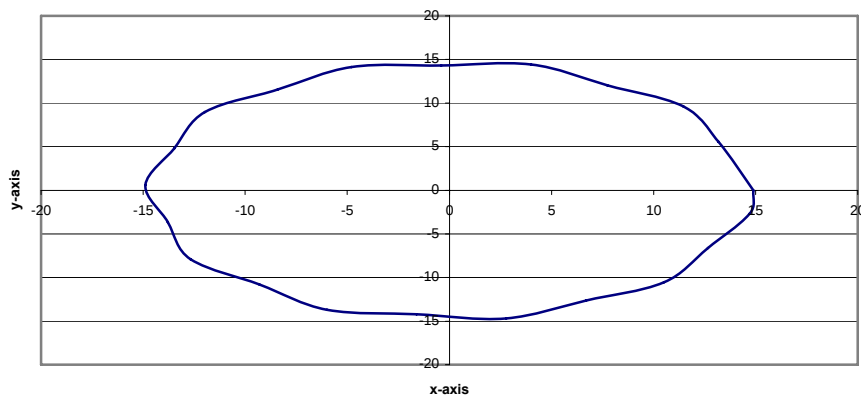


Fig. 1: Plot of the trajectory of Earth around the Sun.

Application's section provides an immediate application of the modification (MKSL) obtained in preceding section. Yet another facet of MKSL emerges from the fact that the existence of nonzero transverse components of the force is mathematically equivalent to the existence of a third body in the physical system being considered (a problem which is examined in another paper by present authors). An orbit equation has been derived, which incorporates spin employing restricted 3-body problem. Then the trajectory of Earth orbit around spinning Sun can be computed employing the numerical scheme. It is interesting to note that the trajectory of Earth turns out to be closed (ellipse).

Acknowledgement

This work was partially supported by Faculty of Science, University of Karachi under a research grant.

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