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DISTRIBUTION OF THE RATIO OF GENERALIZED ORDER STATISTICS FROM PARETO DISTRIBUTION

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Abstract: This paper deals with the probability density function of the ratio of generalized order statistics. We use the Mellin transform techniques to find the distribution of the ratio Z= X_i/X_j where X_i, X_j (i<j) are the ith and jth generalized order statistics from the Pareto distribution.

Keywords: Generalized order, Mellin transform, Pareto distribution, probability density function.

INTRODUCTION

The distributions of the product and the ratio of two random variables are widely used in many areas of statistical analysis as in the problems of selection and ranking rules. They are also found in the context of life testing and in the closely related problems of reliability. For more refer to David [1981], Ahsanulla [1995], Kamps [1995], Aleem and Pasha [1999], Valerray and Novzorv [2001].

Suppose X (l,n,m,k),…,X (n,n,m,k), (k ≥l, m is a real number), are n generalized order statistics. Their joint pdf $f_{1,\ldots,n}$ $(X_1,\ldots,X_n)$ can be written as (see Kamps [1995]):

$$f_1,...,n(x_1,...,x_n)=k\prod_{j-1}^{n-1}\gamma_j\prod_{i-1}^{n-1}(1-F(x_i))^m f(x_i)(1-F(x_n))^{k-1}f(x_n) \quad (1.1)$$

$$\overline{F}^{\,1}(0) < x_1 < \ldots < x_n < \overline{F}(1),$$
$$= \text{otherwise},$$

Where γ_j = k + (n-j)(m+1), $\overline{F}$ (x) = 1- F(x) and f(x) = $\frac{dF(x)}{dx}$.

The joint distribution of ith and jth generalized order statistics is given as:

$$f_{i,j,n,m,k}^{(x,y)}=\frac{c_j}{(i-1)!(j-i-1)!}\left[1-F(x_i)\right]^m\left[g_m(F(x_i))\right]^{i-1}\left[g_m(F(x_j))-g_m(F(x_i)\right]^{j-i-1}$$
$$\left[1-F(x_j)\right]^{\gamma_j-1}f(x_i)f(x_j) \quad (1.2)$$

where $c_j=\prod_{r=1}^{j}\gamma_r$, γ_r = k + (n-r) (m+1)

-∞ < x_j < x_i < ∞ , 0 < i< j ≤ n, k ≥ 1, m is a half number and

$$g_m(x) = \frac{1}{(m+1)}\left[1-(1-x)^{m+1}\right], \; m \neq -1$$
$$= -\log(1-x), \quad m = -1,\ x\mathrm{e}(0,1)$$

Now since

$$\lim_{m \to -1} \frac{1}{(m+1)}\left[1-(1-x)^{m+1}\right] = -\log(1-x),$$

we will write

$$g_m(x) = \frac{1}{(m+1)}\left[1-(1-x)^{m+1}\right] \text{ for all } x \in (0,1)$$

and for all with

$$g^{-1}(x) = \lim_{m \to -1} g_m(x)$$

In this paper we use the Mellin transform technique to find the distribution of the quotient of any two generalized order statistics from the Pareto distribution. We also give practical sub cases which include the distribution of the quotient of the extreme generalized order statistics and that of consecutive generalized order statistics. This ratio is useful in ranking and selection problems. Another special case comprises the peak to median ratio [Morrison and Tobiss 1965], a widely used measure in several sectors of the engineering field, especially in design and communications.

THE MELLIN TRANSFORM

Epstein [1948] was the first to suggest a systematic approach to the study of products and quotients of independent random variables by using a Mellin transform technique. Later, Fox [1957] extended this integral transform to the two dimensional case in order to evaluate products and ratios of random variables x,y with pdf, f(x,y) which is non-negative in the first quadrant and zero else where. The Mellin transform of f(x,y) is defined as:

$$M(s_1, s_2) = \int_0^\infty \int_0^\infty x^{s_1-1} y^{s_2-1} f(x,y)\,dxdy \qquad (2.1)$$

Where s_1 and s_2 are complex variables. Under suitable conditions discussed by Fox [1957] it possesses the inverse.

$$f(x,y) = \frac{1}{(2\pi 1)^2} \int_{h-i\infty}^{h+i\infty} \int_{k-i\infty}^{k+i\infty} M(s_1.s_2) x^{-s_1} y^{-s_2} ds_1 ds_2$$

with the paths of integration being two lines parallel to the imaginary axis and to the right of the origin in the Argand plane.

Extensions when f(x,y) is positive in all four quadrants are stated in Fox [1957] but are not required in this paper. However we will be greatly interested in two particular cases of the above:

$$M(t \mid u) = M(t, t) \qquad (2.2)$$

$$M(t \mid z) = M(t, -t+2) \qquad (2.3)$$

They correspond to the Mellin transforms for the pdf of the product u = x y and the pdf of the ratio z = x / y.

We readily see then that the Mellin transform provides us with a powerful tool in reaching our objective. It enables us to find the distribution of the product and quotient of jointly distributed variants when a mere transformation is either awkward or futile. It also displays its strength in simplify the treatment of probability density function of order statistics. A problem arises however when the inverse of the Mellin transform cannot be found among the formulae in Erdelyi *et al.* [1954], whereupon we must revert to an appropriate transformation.

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The Pareto distribution has the pdf.

$$f(x) = va^{v} x^{-(v+1)} \tag{3.1}$$

With cdf given as

$$F(x) = 1-\left[\frac{x}{a}\right]^{-v} \tag{3.2}$$

Where $x \geq a$ and $a, v > 0$. Now consider two generalized order statistics x_i and y_j the ith and jth respectively with $i < j$, based on a random sample of size n from this distribution. Then by Eq. (1.2), the joint distribution of these generalized order statistics is given by

$$f_{i,j,n,m,k}^{(x,y)} = \frac{c_j v^2 (a)^{v(m+1)(j-i-1)+v(\gamma_j+1)}}{(i-1)!(j-i-1)!(m+1)^{j-2}}\left[1-\left(\frac{x}{a}\right)^{-v(m+1)}\right]^{i-1}\left[(x)^{-v(m+1)}-(y)^{-v(m+1)}\right]^{j-i-1}$$
$$x^{-(v+1)} y^{-(v\gamma_j+1)} \tag{3.3}$$

where $x \leq y$, $x \geq a$, $0 < i < j \leq n$, $a > 0$, $v > 0$, $k \geq 1$, m is a real number. We apply the binomial expansion to Eq. (3.3) we have the above reducing to:

$$f_{i,j,n,m,k}^{(x,y)} = \frac{c_j v^2 a^{v(m+1)(j-i-1)+\gamma_j(v+1)}}{(i-1)!(j-i-1)!(m+1)^{j-2}}\sum_{r=0}^{j-i-1}(-1)^r\binom{j-i-1}{r}x^{-v(m+1)(j-i-r-1)-(v+1)}y^{-v(m+1)r-(v\gamma_j+1)}$$
$$\left[1-a^{v(m+1)}x^{-v(m+1)}\right]^{i-1} \tag{3.4}$$

The Mellin transform of Eq. (3.4) is given by

$$M\left(s_1, s_2\right)=\frac{c_j v^2 a^{v(m+1)(j-i-1)+\gamma_j(v+1)}}{(i-1)!\,(j-i-1)!\,(m+1)^{j-2}}\sum_{r=0}^{j-i-1}(-1)^r\binom{j-i-1}{r}$$
$$\int_a^{\infty}\int_x^{\infty} x^{-v(m+1)(j-i-r-1)-(v+2)+s_1} y^{-v(m+1)r+(v\gamma_j+2)+s_2}\left[1-a^{v(m+1)}x^{-v(m+1)}\right]^{i-1} dydx$$

$$M(s_1,s_2)=\frac{c_j v a^{s_1+s_2-2}}{(i-1)!(j-i-1)!(m+1)^{j-1}}\sum_{r=0}^{j-i-1}(-1)^r\binom{j-i-1}{r}\frac{\beta\left(i, j-i-1+\frac{\gamma_j+1}{(m+1)}-\frac{s_1+s_2-2}{v(m+1)}\right)}{v(m+1)r+v\gamma_j-s_2+1} \tag{3.5}$$

where β(a,b) is the usual beta function. If we now set $s_1 = t$ and $s_2 = -t + 2$ in Eq. (3.5), we obtain by Eq. (3.6) Mellin transform of the distribution of ratio Z of the ith and jth generalized order statistics as:

$$M(t\,|\,z)=\frac{c_j v}{(i-1)!(j-i-1)!(m+1)^{j-1}}\sum_{r=0}^{j-i-1}(-1)^r\binom{j-i-1}{r}\frac{\beta\left(i,j-i-1+\frac{\gamma_j+1}{m+1}\right)}{\gamma(n-j+r+1)+t-1} \quad (3.6)$$

The inverse Mellin transform of Eq. (3.6) as can be seen from Erdelyi *et al*. [1954] is

$$g_{i,j,n,m,k}{}^{(z)}=\frac{c_j v}{(i-1)!(j-i-1)!(m+1)^{j-1}}\beta\left(i,j-i-1+\frac{\gamma_j+1}{m+1}\right)z^{v(n-j+1)-1}\sum_{r=0}^{j-i-1}(-1)^r\binom{j-i-1}{r}z^{vr}$$

$$=\frac{c_j v\beta\left(i,\ j-i-1+\frac{\gamma_j+1}{m+1}\right)}{(i-1)!\,(j-i-1)!\,(m+1)^{j-1}}z^{v(n-j+1)-1}\left(1-z^v\right)^{j-i-1} \quad (3.7)$$

where 0 ≤ z ≤ 1, 0 < i < j ≤ n, γ > 0, k ≥ 1, v > 0, m is a real number.

SPECIFIC RESULTS

(i) If we put i = 1, j = n in Eq. (3.7) we get the pdf of the ratio of the extreme generalized order statistics.

$$g_{1,n,n,m,k}(z)=\frac{c_n v}{(n-1)!(m+1)^{n-1}}\beta\left(1,n-2+\frac{k+1}{m+1}\right)z^{v-1}\left(1-z^v\right)^{n-2} \quad (3.8)$$

where 0 ≤ z ≤ 1 and v > 0, n > 1, k ≥ 1, m is real number.

(ii) If we put j = I+1 in Eq. (3.7), we get the pdf of the ratio of consecutive generalized order statistics as:

$$g_{i,i+1,n,m,k}(z)=\frac{c_{i+1}v}{(i-1)!\,(m+1)^i}\beta\left(i,\frac{\gamma_{i+1}+1}{m+1}\right)z^{v(n-i)-1} \quad (3.9)$$

(iii) If we take n to be odd and putting n = j = 2p + 1, i = p + 1 in Eq. (3.7), we get the pdf of the ratio of peak to median of a random sample of size 2p + 1 generalized order statistics as:

$$g_{p+1,2p+1,2p+1,m,k}(z)=\frac{c_{2p+1}v}{(p)!(p-1)!(m+1)^{2p}}\beta\left(p+1,p-1+\frac{k+1}{m+1}\right)z^{v-1}\left(1-z^v\right)^{p-1} \quad (3.10)$$

From Eq. (3.8), we get the pdf of the ratio of the extreme ordinary order statistics. Moreover Eq. (3.9) gives the pdf of the ratio of the consecutive ordinary order statistics. Finally from Eq. (3.10) we get the pdf of the peak to median ratio of a random sample of size (2p+1) from the Pareto density.

REMARKS

The results of Z^{-1} for m=0 and k=1 in Eqs. (3.7), (3.8), (3.9) and (3.10) are same as given by Aleem and Pasha [1999].

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