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AUTOMORPHISM GROUP GENERATORS FOR FISHER AND YATES 6X6 REDUCED LATIN SQUARES AND THEIR EQUIVALENCE WITH SCHONHARDT'S SPECIES

Asghar Ali
Department of Statistics, Bahauddin Zakariya University, Multan.

Abstract: Fisher and Yates established twenty-two transformation sets for 6x6 Latin squares in 1934, which have been shown to belong to 12 adjugate classes. Schonhardt also established 12 species for this size of Latin squares in 1930. In this paper automorphism group generators have been established for Fisher and Yates transformation sets. Equivalence between the adjugate classes of Fisher and Yates and 12 species of Schonhardt has also been shown.

Keywords: Adjugate squares, automorphism group, Graeco-Latin square, Latin squares, transformation set.

INTRODUCTION

Euler's problem of 36 officers still exists unsolved even after 225 years. It states that the officers were chosen, six from each of the six different regiments, so that the selection from each regiment included one officer from each of the six ranks. The problem was whether it was possible for the officers to parade in six-by-six formation, such that each row and each column contained one member of each rank and one from each regiment. In the terminology of Fisher and Yates [1934,1938], it is a problem of constructing 6x6 Graeco-Latin square. Euler [1782] conjectured and Tarry [1901] proved that no 6x6 Graeco-Latin Square could exist, which means that the officers' problem would remain unsolved. The other order for which Graeco-Latin square cannot exit is the order 2x2. Six-by-six layout is highly useful for orchard experimentation and can be used to test the significance of a variety of sets of treatments. For example, if one has to accommodate two blocking system, one representing the rows and other representing the columns and designs are to be constructed in a 6x6 layout for two non interacting sets treatments; there are various possibilities for which totally balanced designs exist. Ali [1980, 2000], Ali and Clarke [1993], Clarke [1967], Freeman [1964], and Preece *et al.* [1994] have considered such situation and have proposed totally balanced designs for different sizes of the sets of treatments. In this paper, we find all the generators for automorphism groups of 22 transformation sets of 6x6 Latin squares given by Fisher and Yates [1938]. Schonhardt [1930] have also classified of 6x6 Latin squares into 12 species. The correspondence between Fisher and Yates' 22 transformation sets and Schonhardt 12 species has been established.

TERMINOLOGY AND DEFINITIONS

LATIN SQUARE DESIGN

An nxn Latin square design is defined as the arrangement of a set of n treatments in an experimental area divided in n rows and n columns such that each treatment must appear once in a row and once in a column. Treatments are denoted by Latin letters A, B, C,…. If the letters appear alphabetically in first row and first column in a square, such a square is called a standard or reduced Latin square.

GRAECO-LATIN SQUARE

A Graeco-Latin square is an arrangement of a set of n treatments denoted by Greek letters (α, β, γ, η, ….) in a nxn Latin square such that each Greek letter must appear once in each row, once in each column and once with each Latin letter.

TRANSFORMATION SET

Any permutation of the rows and of the columns, or of the letters, or any combination of such permutations called as permutation triplet is known as transformation of a Latin square. Transformation set of Latin squares is such that any transformation of any its members generate a member and such that use of all possible transformations of any member generates all members. Our concern is with 6x6 Latin squares which have 22 transformation sets.

AUTOMORPHISM GROUP

If some permutation triplets are applied on a Latin square from a transformation set and the square remains unaltered, such permutation triplets will constitute an automorphism group.

TOTALLY BALANCED DESIGNS

The designs with the same precision for all treatment comparisons (with equal number of treatments for each comparison) and with equal replications of each treatment are said to be balanced designs.

ADJUGATE SQUARES

The interchange of rows and letters or of columns and letters of a Latin square will generate a new square which will be called adjugate square. The squares obtained from the interchange of rows and columns are called conjugate squares.

GENERATORS FOR AUTOMORPHISM GROUPS OF 22 TRANSFORMATION SETS OF 6X6 LATIN SQUARES

Fisher and Yates [1934] has classified the 6x6 Latin squares into 22 transformation sets and have been given in 17 classes shown in Table 1. Five of these classes numbering II, VI, IX, XII and XVI are in the same adjugate class as their immediate predecessors hence 6x6 Latin squares are classified into 12 adjugate classes. The greatest number of reduced squares found in any set is 1080. If every transformation gave a different square then there would be $6^2 . 5! = 4320$ reduced squares. Using the number of reduced squares for each set of the 22 transformation sets given in Table 1, we can easily calculate the sizes of the automorphism groups, which have been shown in Table 2 along with their generators.

Table 1: Transformation Sets of 6x6 Latin Squares prepared by Fisher and Yates [1934].

I					
A	B	C	D	E	F
B	C	F	A	D	E
C	F	B	E	A	D
D	E	A	B	F	C
E	A	D	F	C	B
F	D	E	C	B	A
0001-1080 1081-2I60					

II					
A	B	C	D	E	F
B	C	F	E	A	D
C	F	B	A	D	E
D	E	A	B	F	C
E	A	D	F	C	B
F	D	E	C	B	A
2161-3240					

III					
A	B	C	D	E	F
B	A	F	E	C	D
C	F	B	A	D	E
D	C	E	B	F	A
E	D	A	F	B	C
F	E	D	C	A	B
2341-4320					

IV					
A	B	C	D	E	F
B	A	E	F	C	D
C	F	B	A	D	E
D	E	A	B	F	C
E	D	F	C	B	A
F	C	D	E	A	B
4321-5400					

V					
A	B	C	D	E	F
B	A	E	C	F	D
C	F	B	A	D	E
D	E	F	B	C	A
E	D	A	F	B	C
F	C	D	E	A	B
5401-5940 5941-6480					

VI					
A	B	C	D	E	F
B	A	F	E	C	D
C	F	B	A	D	E
D	E	A	B	F	C
E	C	D	F	B	A
F	D	E	C	A	B
6481-7020					

VII					
A	B	C	D	E	F
B	C	D	E	F	A
C	E	A	F	B	D
D	F	B	A	C	E
E	D	F	B	A	C
F	A	E	C	D	B
7201-7560					

VIII					
A	B	C	D	E	F
B	A	E	F	C	D
C	F	A	E	D	B
D	C	B	A	F	E
E	D	F	C	B	A
F	E	D	B	A	C
7561-7560 7921-8280					

IX					
A	B	C	D	E	F
B	A	E	F	C	D
C	F	A	E	D	B
D	E	B	A	F	E
E	D	F	C	B	A
F	C	D	B	A	C
8281-8640					

X					
A	B	C	D	E	F
B	C	F	A	D	E
C	F	B	E	A	D
D	A	E	B	F	C
E	D	A	F	C	B
F	E	D	C	B	A
8641-8820					

XI					
A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	F	E	B	A	C
E	D	F	C	B	A
F	E	D	A	C	B
8821-8940 8941-9060					

XII					
A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	E	F	A	B	C
E	F	D	C	A	B
F	D	E	B	C	A
9061-9180					

XIII					
A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	F	E	B	A	C
E	D	F	A	C	B
F	E	D	C	B	A
9181-9240					

XIV					
A	B	C	D	E	F
B	C	A	E	F	D
C	A	B	F	D	E
D	F	E	B	A	C
E	D	F	C	B	A
F	E	D	A	C	B
9241-9280					

XV					
A	B	C	D	E	F
B	A	F	E	D	C
C	D	A	B	F	E
D	F	E	A	C	B
E	C	B	F	A	D
F	E	D	C	B	A
9281-9310 93I7-9352					

XVI					
A	B	C	D	E	F
B	A	E	C	F	D
C	E	A	F	D	B
D	C	F	A	B	E
E	F	D	B	A	C
F	D	B	E	C	A
9353-9388					

XVII					
A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	E	F	A	B	C
E	F	D	C	A	B
F	D	E	B	C	A
9389-94o8					

Automorphism group size for the transformation set I is shown as four, which can easily be justified by manipulating the group generators shown in Table 2. The first automorphism group denoted by π_1 is obtained by interchanging the symbols C with D and D with E; by interchanging the rows 2 with 3 and 4 with 5 and by interchanging the columns 2 with 3 and 4 with 5. The second automorphism group denoted by π_2 is obtained by interchanging the symbols A with F, B with D and C with E; by interchanging the rows 1 with 6, 2 with 4 and 3 with 5 and by interchanging the 4 with 5.

Table 2: Automorphism Group Size and Their Group Generators For 22 Transformation Sets of 6x6 Latin Squares.

Transformation sets	Group Generators				
	Symbols	Rows	Columns	No. Reduced Squares	Automorphism group size
I	(BC)(DE) (AF)(BD)(CE)	(23)(45) (16)(24)(35)	(23)(45) (45)	1080	4
I´	(BC)(DF) (AF)(BD)(CE)	(23)(45) (45)	(23)(45) (16)(24)(35)	1080	4
II	(BC)(DE) (BC)	(23)(45) (16)(24)(35)	(23)(45) (16)(24)(35)	1080	4
III	(AD)(BC)(EF) (AF)(DE)	(26)(34) (15)(24)(36)	(14)(23)(56) (14)(25)(36)	1080	4
IV	(CD)(EF) (CE)(DF)	(34)(56) (35)(46)	(34)(56) (35)(46)	1080	4
V	(CEFD) (AB)(CF)	(3564) (35)(46)	(3564) (35)(46)	540	8
V´	(CEFD) (AB)(CF)	(3564) (35)(46)	(3564) (35)(46)	540	8
VI	(CEDF) (CE)(DF)	(3546) (12)(34)	(3546) (12)(34)	540	8
VII	(AB)(DE) (AE)(BD)(CF) —	(15)(24)(36) (16)(35) (13)(24)(56)	(14)(25)(36) (13)(24)(56) (13)(25)(46)	540	8
VIII	(BC)(EF) (AFCDBE)	(23)(56) (243)	(23)(56) (163425)	360	12
VIII´	(BC)(EF) (AFCDBE)	(23)(56) (163425)	(23)(56) (243)	360	12
IX	(CD)(EF) (BCD)	(34)(56) (153246)	(34)(56) (163245)	360	12
X	(BC)(DE) — (AECFBD)	(23)(45) (16)(25)(34) (135264)	(23)(45) (16)(25)(34) (145)(263)	180	24
XI	(AFCEBD) (AFBECD)	(12)(465) (45)	(143625) (162534)	120	36
XI´	(AFCEBD) (AFBECD)	(143625) (162534)	(12)(465) (45)	120	36
XII	(BC) (ACB)(DE)	(152634) (143526)	(143625) (143526)	120	36
XIII	(AB)(EF) — (BC)(DE)	(142635) (142635) (23)(45)	— (153624) (23)(45)	60	72
XIV	(AFBDCE) (AEBDCF) (ABC)	(162435) (12)(45) (123)	(23)(45) (142635) (465)	40	108
XV	(AEFDBC) (BCDEF) (BCED)	(246)(35) (23456) (2354)	(156423) (23456) (2354)	36	120
XV´	(AEFDBC) (BCDEF) (BCED)	(156423) (23456) (2354)	(246)(35) (23456) (2354)	36	120
XVI	(BDF)(CE) (BCDEF) (BCED)	(156423) (23456) (2354)	(156423) (23456) (2354)	36	120
XVII	— — (AECFBD) (ADBECF)	(123)(456) (14)(26)(35) (456) (23)(46)	(132)(456) (14)(25)(36) (153624) (142536)	20	216

Composing permutations, we find that applying initially π_1 and then π_2 gives the third automorphism group $\pi_3 = \pi_1\ \pi_2 = \pi_2\ \pi_1$, which is described as:

[(BC)(DE)][(AF)(BD)(CE)] = (AF)(BE)(CD) for symbols,
[(23)(45)][(16)(24)(35)] = (16)(25)(34) for rows and
[(23)(45)][(45)] = (23) for columns.

The fourth automorphism group is denoted by $\pi_0 = I$, which is the identity permutation of the triplet i.e. the Latin square is used in its original reduced form. The automorphism group sizes for the transformation sets I´, II, III and IV can easily be confirmed and corresponding permutation triple can be obtained by applying the above used technique.

Automorphism group size is shown as 8 for the transformation set V and two automorphism groups are given in form of the permutation triple in Table 2. If these are denoted by π_1 and π_2 respectively, the remaining automorphism groups are obtained as:

$$\pi_3= \pi_1 \pi_2,\ \ \pi_4= \pi_2 \pi_1,\ \ \pi_5= \pi_3 \pi_1,\ \ \pi_6= \pi_4 \pi_2,\ \ \pi_7= \pi_3 \pi_4,\ \ \pi_8= \pi_0 = I$$

In the same way, the given number of automorphism groups can be obtained for each of the remaining transformation sets.

CORRESPONDENCE BETWEEN FISHER AND YATES' 22 TRANSFORMATION SETS AND SCHONHARDT'S 12 SPECIES

It has already been stated that Fisher and Yates [1934] have classified 6x6 Latin Squares into 12 adjugate classes. The equivalence of 12 species for 6x6 Latin squares as given in Table 3 established by Schonhardt [1930] with the above said 12 adjugate classes is shown in Table 4. Permutation triple has been to Schonhardt's proposed species to obtained the corresponding Fisher and Yates' adjugate classes.

Table 3: 12 Species of 6x6 Latin Squares Established by Schonhardt [1930].

I						II						III						IV					
A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F
B	C	D	E	F	A	B	A	E	F	C	D	B	C	A	E	F	D	B	A	D	C	F	E
C	D	E	F	A	B	C	F	A	E	D	B	C	A	B	F	D	E	C	D	E	F	A	B
D	E	F	A	B	C	D	E	F	A	B	C	D	F	E	B	A	C	D	C	F	E	B	A
E	F	A	B	C	D	E	D	B	C	F	A	E	D	F	C	B	A	E	F	A	B	D	C
F	A	B	C	D	E	F	C	D	B	A	E	F	E	D	A	C	B	F	E	B	A	C	D
V						VI						VII						VIII					
A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F
B	A	E	F	C	D	B	A	D	E	F	C	B	A	D	C	F	D	B	A	F	E	C	D
C	F	B	E	D	A	C	F	B	A	D	E	C	E	A	F	D	B	C	F	A	B	D	E
D	E	F	B	A	C	D	E	F	B	C	A	D	F	E	A	B	C	D	E	B	A	F	C
E	D	A	C	F	B	E	C	A	F	B	D	E	C	F	B	A	D	E	C	D	F	A	B
F	C	D	A	B	E	F	D	E	C	A	B	F	D	B	E	C	A	F	D	B	C	E	A
XI						X						XI						XII					
A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F	A	B	C	D	E	F
B	C	A	F	D	E	B	A	F	E	D	C	B	A	D	E	F	C	B	A	F	E	D	C
C	A	B	E	F	D	C	E	A	B	F	D	C	D	B	F	A	C	C	E	D	B	F	A
D	F	E	A	B	C	D	F	B	A	C	E	D	E	F	B	C	A	D	F	B	C	A	E
E	D	F	B	C	A	E	C	D	F	B	A	E	F	A	C	B	D	E	D	F	A	C	B
F	E	D	C	A	B	F	D	E	C	A	B	F	C	E	D	A	B	F	C	A	E	B	D

CONCLUSIONS

As already specified that no Graeco Latin square exits for a 6x6 layout, but many other possibilities of construction of totally balanced design for a single set of treatments and two non-interacting sets of treatments of the multiple sizes of 36 can successfully be considered. The automorphism group generators for Fisher and Yates (1934)[1] transformation sets of 6x6 Latin squares along with their correspondence with Schonhardt's 12 species can be useful for the establishment of methods of construction of totally balanced designs for this layout.

Table 4: Equivalence between Fisher and Yates Adjugate Classes and Schonhardt's Species for 6x6 Latin Squares.

Schonhardt's Species	Fisher and Yates' Adjugate Classes	Symbols	Rows	Columns
XII	II	(BFEDC)	(26543)	(26543)
VI	III	(CD)(EF)	(34)(56)	(34)(56)
VII	IV	(CFE)	(12)(456)	(12)(346)
XI	VI	(DE)	(12)(356)	(12)(356)
V	VII	(ACDEBF)	(12435)	(125)
X	IX	(CD)	(34)	(56)
IV	X	(EF)	(56)	(56)
IX	XII	(DF)	(123)(45)	(132)(45)
I	XIII(b)			
III	XIV			
VIII	XVI	(DFE)	(465)	(465)
II	XVII	(BFCDE)	(26345)	(26345)

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