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NUMBER OF PERIODIC SOLUTIONS OF CLASSES $C_{2,4}$ AND $C_{4,3}$

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Abstract: The classes $C_{2,4}$ and $C_{4,,3}$ of coefficients of one-dimensional non-autonomous differential equations are investigated for periodic solutions. Seven and Five periodic solutions are found for classes $C_{2,4}$ and $C_{4,3}$ respectively. The method we used is bifurcation of periodic solution from a fine - focus.

Keywords: Multiplicity, periodic solution, transformation.

INTRODUCTION

This paper is concerned with solutions of non-autonomous equations of the form:

$$\dot{z} = \alpha(t)\, z^3 + \beta(t)\, z^2 + \gamma\, z + \delta \qquad \text{(A)}$$

where γ and δ are small, $\alpha(t)$ and $\beta(t)$ are polynomials in t. it is one of the first order non-autonomous differential equations that have attracted the interest of the mathematicians during the last decades. Apart from the fact that (A) frequently appears in the applications, it is studied because many other systems can be transformed into this form, see for instance Al-wash and Lloyd [1987], Yasmin [2001] and Lloyd [1973].

By one side a large no of criteria for non-existence, existence, uniqueness of periodic solutions have been found [Neto 1980, Shi-Songling 2001]. On the other side, after fixing some classes of (A) lower bounds for its number of periodic solutions is given. These bounds are obtained either by perturbing fine foci [Al-wash and Lloyd 1987, Lloyd 1986]. Our results include some of the previous ones obtained by Al-wash and Lloyd [1987]. Our technique is based on a nice theorem of these last authors, which we enunciate in the form of theorem 2.1 after some preliminary definitions.

Remember that a given system of the form

$$dx/dt = \lambda\, x + y + p(x, y)$$
$$dy/dt = -x + \lambda\, y + q(x, y) \qquad (1.1)$$

where p and q are homogeneous polynomial of degree n, the origin is then a fine focus if $\lambda = 0$. Such systems can be transformed to non-autonomous equations:

$$d\rho/d\theta = \alpha(\theta)\, \rho^3 + \beta(\theta)\, \rho^2 - \lambda\, (n-1)\, \rho \qquad (1.2)$$

where $\rho = r^{n-1}\, (1 - r^{n-1}\, g(\theta))^{-1}$ (1.3)

and $\alpha(\theta) = -(n-1)\, g(\theta)\, \{ f(\theta) + \lambda\, g(\theta) \}$

$$\beta(\theta) = -(n-1)\, \{f(\theta) + 2\lambda\, g(\theta)\} + g'(\theta) \qquad (1.4)$$

Thus $\alpha(\theta)$ and $\beta(\theta)$ are homogenous polynomials in $\cos\theta$ and $\sin\theta$ of degree 2(n+1) and (n+1) respectively.

The relationship between (1.1) and (1.2) is explained by Al-wash and Lloyd [1987]. For completeness we give here few details. The transformation (1.3) is defined in

$$D = \{(r,\theta);\ r^{n-1}\, g(\theta) < 1\} \qquad (1.5)$$

which is an open set containing the origin. It is clear that the limit cycles (an isolated closed orbits) of (1.1) correspond to positive 2π-periodic solutions of (1.2) with ρ small and positive. Al-wash and Lloyd [1987] have given examples, which demonstrate that there is no upper bound for the number of periodic solutions of (1.2) unless the coefficients are suitably restricted. Thus the real question therefore is the maximum possible number of periodic solutions for various classes of coefficients, which is part of the list of problems presented in Paris by Hilbert [1902]. Therefore in this paper we consider different classes. In order to obtain good estimates of number of periodic solutions of any class of differential equation we use complexified form [Bautin 1954, Lloyd 1973]:

$$\dot{z} = \alpha(t)\, z^3 + \beta(t) z^2 + \gamma(t) z \qquad (1.6)$$

where z is complex but t remains real and the coefficients α, β, γ are real valued functions. We consider the classes $C_{2,4}$ and $C_{4,3}$ to investigate the possible number of 2π periodic solutions where $C_{n1,n2}$ denotes the class of equation of the form (1.6) in which α is of degree n_1 and β is of degree n_2. These classes were investigated by Al-wash and Lloyd [1987], and Yasmin [2001] for different n_1 and n_2. Still it is an open question, what is an upper bound for the number of 2π -periodic solutions of such classes. Thus for each class we calculate the maximum possible multiplicity of the origin $z = 0$. We specify $\omega \varepsilon R$ and seek information about the number of solutions, which satisfy the periodic boundary condition

$$z(0) = z(\omega)$$

We say such solutions periodic whether or not α, β, γ themselves periodic. We are particularly interested in the case when $\omega = 1$ and α, β, γ are of form arising in Al-wash and Lloyd [1987], and Shi-Songling [1975]. The multiplicity of $z = 0$ as a solution of (1.6) is the multiplicity of $z = 0$, as a zero of the displacement function

$$q : c \rightarrow z(\omega,0,c) - c$$

To compute the multiplicity (which we call μ), we write

$$z(t;0,c) = \sum a_n(t)\, c^n$$

where $0 \leq t \leq \omega$ and c in a neighborhood of 0, and substitute directly into the equation. This gives a recursive set of linear differential equations for $a_n(t)$; the initial conditions are

$$a_1(0) = 1,\ a_j(0) = 0 \text{ for } j > 1$$

It can be seen that

$$\dot{a}_1(t) = a_1(t)\, \gamma(t)$$

where $a_1(t) = \exp [\int_0^w \gamma (s)\, ds].$

Thus, $\mu > 1 \text{ iff } [\int_0^w \gamma (s)\, ds = 0$ (1.7)

Since we are interested in the case when z = 0 is a multiple solution, we shall assume that (1.7) holds. Using the transformation

$$\xi = z \exp [-\int_0^w \gamma (s)\, ds]$$

Then equation (1.6) becomes

$$\dot{\xi} = \hat{\alpha}(t)\xi^3 + \hat{\beta}(t)\xi^2 \quad (1.8)$$

where $\hat{\alpha}(t) = \alpha(t) \exp [2\int_0^w \gamma (s)\, ds]$

and $\hat{\beta}(t) = \beta(t) \exp [\int_0^w \gamma (s)\, ds]$

We note that if the function α, β and γ are periodic, then so are $\hat{\alpha}$ and $\hat{\beta}$. Also if multiplicity of z = 0 as a periodic solution of (1.6) is $\mu > 1$, then the multiplicity of z = 0 as a periodic solution of (1.8) is μ. We therefore suppose that $\gamma(t) \equiv 0$ that is $\lambda = 0$ in (1.6), and so we consider equations of the form

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2 \quad (1.9)$$

For this $a_1(t) \equiv 1$ and for n > 1, functions $a_n(t)$ are determined by the relation:

$$\dot{a}_n = \alpha \sum_{\substack{i+j+k=n \\ i,j,k\geq 1}} a_l\, a_j\, a_k + \beta \sum_{\substack{i+j=n \\ i,j\geq 1}} a_i\, a_j \quad (1.10)$$

These equations can be solved recursively but their calculation is complicated involving integration by parts. Let $\eta_i = a_i(\omega)$; then multiplicity μ of z = 0 is k if $\eta_1=1$, $\eta_2 = \ldots = \eta_{k-1} = 0$ but $n_k \neq 0$. These η_i are called focal values. The formulae for $a_k(t)$ and η_k for $k \leq 9$ are given by Al-wash and Lloyd [1987], and Yasmin [2001]. We are presenting η_k for $k \leq 9$ in the next section to calculate the focal values. For the calculation of these focal values we have used a computer package “DERIVE” a mathematical assistant to perform algebraic operations accurately. The calculations in this paper are verified by using "DERIVE". At each stage of calculation the degree of polynomial involved becomes higher and higher, so it cannot be solved manually. In proceeding sections, conditions for z=0 to be a center and periodic solution of classes $C_{2,4}$ and $C_{4,3}$ are presented.

CALCULATION OF MULTIPLICITY OF z = 0 AND METHOD OF PERTURBATION

Calculation of multiplicity involve definite integral of type $\int \alpha(t)\ \bar{\beta}(t)\, dt$. We use a bar over a function to denote its definite integral

$$\bar{\beta}(t) = \int_o^{\omega} \beta(t)\, dt.$$

THEOREM 2.1

The solution z = 0 of (1.9) has multiplicity k where $2 \le k \le 9$ if and only if $\eta_i = 0$ for $2 \le i \le k-1$ and $\eta_k \ne 0$ where η_k are given by:

$$\eta_2 = \int_0^{\omega} \beta$$

$$\eta_3 = \int_0^{\omega} \alpha$$

$$\eta_4 = \int_0^{\omega} \alpha\bar{\beta}$$

$$\eta_5 = \int_0^{\omega} \alpha\bar{\beta}^2$$

$$\eta_6 = \int_0^{\omega} (\alpha\bar{\beta}^3 - \tfrac{1}{2}\,\bar{\alpha}^2\beta)$$

$$\eta_7 = \int_0^{\omega} \alpha\bar{\beta}^4 + 2\,\alpha\,\bar{\alpha}\,\bar{\beta}^2$$

$$\eta_8 = \int_0^{\omega} \alpha\bar{\beta}^5 + 3\,\alpha\,\bar{\alpha}\,\bar{\beta}^3 + \alpha\bar{\beta}^2\,\overline{\bar{\beta}\alpha} - \tfrac{1}{2}\,\bar{\alpha}^3\beta$$

$$\eta_9 = \int_0^{\omega} \alpha\bar{\beta}^6 - 5\,\alpha\,\bar{\alpha}\,\bar{\beta}^4 - 2\,\bar{\beta}^3\,\overline{\beta\bar{\alpha}} + 20\,\overline{\beta\bar{\alpha}^2} + 2\,\overline{\beta\bar{\alpha}}\,\beta\bar{\alpha}^2$$

Having determined the multiplicity μ, the aim will be to construct equations with maximum possible number of distinct real periodic solutions. The idea is to make a sequence of perturbations of the coefficients, each of which causes one periodic solution to bifurcate out of the origin. The method used is as follows:

We start with an equation of the form (1.9) for which μ = k (say). Let U′ be a neighborhood of the origin in the complex plane containing no periodic solutions other than z = 0. Then by theorem (2.4) of Al-wash and Lloyd [1987] the total number of periodic solutions with initial points in U is unchanged by sufficiently small perturbation of the coefficients. If possible, we perturb the coefficients α, β and γ so that $\eta_2 = \eta_3 = \ldots. = n_{k-2} = 0$ but $n_{k-1} \neq 0$. Then there is a non-trivial periodic solution $\psi(t)$, say, with $\psi(0) \in U'$, and the only periodic solutions in U' are ψ and the zero solution. Since complex solutions occur in conjugate pairs, it follows that ψ is real. Now let W_1 be a neighborhood of ψ and U_1 be a neighborhood of 0 such that $U_1 \cup W_1 \subset U'$ and $U_1 \cap W_1 = \phi$. The number of periodic solutions with initial points in each of U_1 and W_1 is preserved under sufficiently small perturbations of the coefficients. We then seek to perturb the coefficients further such that $\eta_2 = \eta_3 = \ldots. = \eta_{k-3} = 0$, but $n_{k-2} \neq 0$. In this case $\mu = k - 2$. Now a second real non-trivial periodic solution has initial point in U_1; there remains a real periodic solution with initial point in W_1. Thus we have two non-trivial real periodic solutions and the zero solution is of multiplicity k – 2. Continuing in this way we end up with an equation of the form (1.9) with $\mu = 2$ and k – 2 distinct non-trivial real periodic solutions.

CONDITIONS FOR A CENTRE

To find maximum possible value of μ for classes $C_{2,4}$ and $C_{4,3}$ of coefficients, we evaluate the quantities $\eta_k = a_k(\omega)$ which are given in section 2, until a value K of k is found with property that $\eta_k = 0$ for all k if $\eta_2 = \eta_3 = \ldots. = \eta_{k+1} = 0$. Then μ_{max} is the smallest such K. In association with the method which we have described for calculating the η_k, we need conditions which are sufficient for z = 0 to be a centre. Only then we know that we need to calculate no more of the η_k. Now we will state here the conditions for z = 0 to be a centre [Al-wash 1987] because we will need them in the next section.

THEOREM 3.1

Suppose that there is a differentiable function $\sigma(\omega) = \sigma(0)$ and continuous function f and g defined on $I = \sigma([0, \omega)]$ such that

$$\alpha(t) = f(\sigma(t))\ \dot{\sigma}\ ,\ \beta(t) = g(\sigma(t))\ \dot{\sigma}$$

Then origin is a centre for equation

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2$$

COROLLARY 3.2

Suppose that α or β is identically zero and the other has mean value zero, then origin is a centre.

PERIODIC SOLUTIONS OF CLASS $C_{2,4}$ AND $C_{4,3}$

In this section, we shall consider the classes $C_{2,4}$ and $C_{4,3}$ in which $C_{2,4}$ denote the class of equations of the form

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2 \qquad \text{(B)}$$

where $\alpha(t)$ and $\beta(t)$ are polynomial in t of degree 2 and 4 respectively and $C_{4,3}$ denote the class of equation of the form (B) in which $\alpha(t)$ and $\beta(t)$ are polynomial of degree 4 and 3 respectively. Al-wash and Lloyd [1987] proved that for class $C_{k,1}$:

$$\mu_{max}(C_{k,1}) = 4 \text{ for } k = 2, 3$$
$$= 5 \text{ for } k = 4, 5$$

Yasmin [2001] proved

$$\mu_{max}(C_{1,k}) = 4 \text{ for } k = 2, 3$$
$$= 8 \text{ for } k = 4, 5.$$

THEOREM 4.1

Let $C_{2,4}$ denote the class of equations of the form:

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2$$

where $\alpha(t)$ is of degree 2 and $\beta(t)$ is of degree 4.

Then we have $\mu_{max}(C_{2,4}) = 7$.

Proof

Let $\alpha(t) = a + c\,t^2$

$\beta(t) = d + e\,t + f\,t^2 + h\,t^4$

Then by theorem 2.1

$$\eta_2 = d + e/2 + f/3 + h/5 \qquad \text{(i)}$$

and $\eta_3 = a + e/3$ (ii)

Now multiplicity of the origin z = 0 is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu_2 = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If

$$\eta_2 = \eta_3 = 0.$$

Then from (i) and (ii) respectively, we have

$$d = -\frac{15e + 10f + 6h}{30} \qquad \text{(iii)}$$

and $a = -c/3$ (iv)

Using (iii) and (iv), $\alpha(t)$ and $\beta(t)$ become as

$$\alpha(t) = c(t^2 - 1/3)$$

$$\beta(t) = \frac{15e(2t-1) + 2(5f(3t2-1) + 3h(5t4-1)}{30}$$

Then we calculate η_4

$$\eta_4 = c\,(e - h)/360 \qquad \text{(v)}$$

If $\eta_4 = 0$, then either c = 0 or e = h. If c = 0 then by equation (iv) a = 0. Hence $\alpha(t) \equiv 0$ and also mean value of $\beta(t) = 0$, so by corollary (3.2) the origin is a centre. So we take $c \neq 0$. Now if e = h, $c \neq 0$, $\beta(t)$ becomes

$$\beta(t) = \frac{15h(2t-1) + 2(5f(3t2-1) + 3h(5t4-1)}{30}$$

and $\alpha(t) = c(t^2 - 1/3)$

Then theorem 2.1

$$\eta_5 = \frac{ch(1105f + 1908h)}{81081000}$$

If $\eta_5 = 0$, as $c \neq 0$ implies either h = 0

or $$f = -\frac{1908h}{1105} \qquad \text{(vi)}$$

If h = 0, then it becomes $C_{2,2}$. In this case μ_{max} $(C_{2,2}) = 4$. To have multiplicity of origin greater than 4, we suppose $h \neq 0$ and we calculate η_6

$$\eta_6 = \frac{ch(83029700c + 7083h^2)}{12184852680000}$$

Now $\eta_6 = 0$ gives

$$c = -\frac{7083h^2}{83029700} \qquad \text{(vii)}$$

Using (vi) (viii) and theorem (2.1) η_7 becomes

$$\eta_7 = -\frac{189602504563h^2}{399876586378272561558000000}$$

which is nonzero as $h \neq 0$.

Hence μ_{max} $(C_{2,4}) = 7$.

THEOREM 4.2

In the equation $\dot{z} = \alpha(t) z^3 + \beta(t)z^2$ (B)

Let $$\alpha(t) = -\frac{7083h^2(3t^2-1) - 83029700(\in_1(3t^2-1) + 3\in_4))}{249089100}$$

and $$\beta(t) = 3h\,(2210\,t^4 - 3816\,t^2 + 2210\,t - 275) + \frac{1105(2\in_2(3t^2-1) + 3(\in_3(2t-1) + 2\in_5))}{6630}$$

with $h \neq 0$.

If $\in_i$, $1 \leq i \leq 5$ are chosen to be non zero and in that order each $\in_k$ is sufficiently small as compared with $\in_{k-1}$, then equation (B) has five non-trivial real periodic solutions.

Proof

The coefficients are chosen so that the multiplicity of origin, μ is 7 if $\in_i = 0$ for $1 \leq i \leq 5$. Choose $\in_1 \neq 0$ but $\in_i = 0$ for $2 \leq i \leq 5$. Then it can be checked that $\eta_2=\eta_3=\eta_4=\eta_5= 0$ but $\eta_6 \neq 0$. Thus multiplicity of origin is reduced by one.

Therefore one real solution bifurcates out of the origin. Next suppose that $\in_2 \neq 0$ but $\in_3 = \in_4 = \in_5 = 0$. We may check that $\eta_2 = \eta_3 = \eta_4 = 0$ but η_5 is a constant multiple of $\in_2$. So $\mu = 5$. If $\in_2$ is small enough, then there are two non-trivial real periodic solutions continuing in this way, we have five non-trivial real periodic solutions.

COROLLARY 4.3

With $\alpha(t)$ and $\beta(t)$ as in Theorem (4.2) the equation

$$\dot{z} = \alpha(t) z^3 + \beta(t) z^2 + \gamma z + \delta \qquad \text{(C)}$$

has seven real periodic solutions if γ and δ are small.

Proof

If $\gamma = 0$ and $\delta = 0$, $\mu = 0$. Then (C) has five real periodic solutions. If $\gamma \neq 0$ but small enough, then $\mu = 1$ and by the argument used in above theorem there are six distinct real periodic solutions z = 0 is another solution. So we have seven real periodic solutions.

THEOREM 4.4

Let $C_{4,3}$ denote the class of equations of the form:

$$\dot{z} = \alpha(t) z^3 + \beta(t) z^2$$

in which α and β are of degree 4 and 3 respectively.

Then $\quad \mu_{max}(C_{4,3}) = 5.$

Proof

Let $\quad \alpha(t) = a + b(2t-1) + c(2t-1)^2 + d(2t-1)^3 + e(2t-1)^4$

$\quad \beta(t) = f + j(2t-1)^3$

Then by theorem 2.1

$$\eta_2 = f$$

and $\quad \eta_3 = a + (5c + 3e)/15$

Now multiplicity of the origin is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$.

If $\eta_2 = \eta_3 = 0$. Then we have

$$f = 0 \qquad \text{(a)}$$

and $\quad a = -(5e + 3e)/15 \qquad$ (b)

and η_4 becomes

$$\eta_4 = j(15c + 14e)/1575$$

If $\eta_4 = 0$, either $j = 0$ or

$$c = -\frac{14e}{15}.$$

If $j = 0$. Then $\beta(t) = 0$, also mean value of $\alpha(t) = 0$, so by corollary 3.2 origin is a centre. Thus $j \neq 0$.

If $\quad c = -14e/15 \qquad$ (c)

Then $\alpha(t)$ becomes

$$\alpha(t) = b(2t-1) + \frac{45d(2t-1)^3 + 8e(90t^4 - 180t^3 + 114t^2 - 24t + 1)}{45}$$

Then by theorem 2.2

$$\eta_5 = 4\, e\, j^2/57915$$

If $\eta_5 = 0$, as $j \neq 0$ so $e = 0$. If $e = 0$ then by equation (c) $c = 0$.
Thus $\alpha(t)$ and $\beta(t)$ becomes as

$$\alpha(t) = b(2t -1) + d(2t - 1)^3$$
$$\beta(t) = j(2t - 1)^3$$

These can be written as

$$\alpha(t) = \{(b + d(2t - 1)^2 \}(2t - 1)$$
$$\beta(t) = j(2t - 1)^2 (2t - 1)$$

Let $\sigma(t) = t^2 - t$ such that $\sigma(0) = \sigma(1)$. Then $\dot{\sigma}(t) = 2t - 1$

Thus $\alpha(t) = [b + d(4\sigma + 1)]\, \dot{\sigma}$

$\beta(t) = j(4\sigma + 1)\, \dot{\sigma}$

Therefore, $\alpha(t) = f(\sigma(t))\, \dot{\sigma}$

$\beta(t) = g(\sigma(t))\, \dot{\sigma}$

where $f(\sigma) = b + d(4\sigma + 1)$ and $g(\sigma) = j(4\sigma + 1)$

So by theorem 3.1 origin is a centre for the equation

$$\dot{z} = \alpha(t) z^3 + \beta(t)z^2$$

Thus $e \neq 0$.

Hence $\mu_{max}(C_{4,3}) = 5$.

THEOREM 4.5

In the equation

$$\dot{z} = \alpha(t) z^3 + \beta(t) z^2 \qquad \text{(B)}$$

Let

$$\alpha(t) = d(2t-1) - \frac{5c - 15d(2t-1)^3 - 2e(120t^4 - 240t^3 + 152t^2 - 32t - 1) - 15(\in_1)2t-1)^2 + \in_2)}{15}$$

$\beta(t) = j(2t - 1)^3 + \in_3)$

If $\in_3$; $1 \leq i \leq 3$ are chosen to be non-zero and in that order such that each $\in_k$ is sufficiently small as compared with $\in_{k-1}$, then equation (B) has three non-trivial real periodic solutions.

Proof

The coefficients are chosen so that multiplicity of the origin, μ is 5 if $\in_i = 0$ for $0 \leq i \leq 3$. Choose $\in_1 \neq 0$ but $\in_i = 0$ for $2 \leq i \leq 3$ then it can be checked that $\eta_2 = \eta_3 = 0$ but $\eta_4 \neq 0$; thus the multiplicity of the origin is reduced by one. Hence $\mu = 4$. Therefore one real solution can bifurcates out of the origin. Next with $\in_2 = 0$ but $\in_3 \neq 0$, we have $\eta_2 = 0$ but η_3 is a constant multiple of $\in_2$; so $\mu = 3$. If $\in_2$ is small enough there are two real non-trivial periodic solutions. Continuing in this way, we have three non-trivial real periodic solutions.

COROLLARY 4.6

With $\alpha(t)$ and $\beta(t)$ as in theorem 4.5, the equation

$$\dot{z} = \alpha(t)z^3 + \beta(t)z^2 + \gamma z + \delta \qquad \text{(C)}$$

has five distinct real periodic solutions if γ and δ are small enough.

Proof

If $\gamma = 0$, $\delta = 0$, $\mu = 2$ then $\mu = 0$ and equation (C) has three real periodic solution. If γ is nonzero but small enough, then $\mu = 1$ and by the argument used in the above theorem, there are four distinct real periodic solution; z = 0 is another solution.

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