▼ Journal of Research (Science), Bahauddin Zakariya University, Multan, Pakistan.
Vol.12, No.2, December 2001, pp. 189-193 ISSN 1021-1012

ON MULTIPLICATION OF HALL-LITTLEWOOD FUNCTIONS

N. Sultana
Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan.
email: pdnaz@yahoo.com

Abstract: The subject of investigation in this paper is the multiplication of Hall-Littlewood symmetric functions. The main tool is to find the particular formula for the coefficient $f^{\upsilon}_{\lambda,(r-1,r)}$(t) in the product of two HL-functions.

Keywords: HL-functions, horizontal strip, Schur function, Littlewood-Richardson rule.

INTRODUCTION

In this paper we shall discuss the multiplication of Hall-Littlewood functions. For this purpose we need to define the following concepts.

Let λ and μ be partitions of n, such that $\mu_i \le \lambda_i$ (i = 1,…,n). If the diagram [Macdonald 1995] of λ contains the diagram of μ. Then the skew diagram is the set theoretic difference $\theta = \lambda - \mu$. The conjugate of the skew diagram

$$[\theta] = [\lambda - \mu] \text{ is } [\theta'] = [\lambda' - \mu']$$

Let $\theta_i = \lambda_i - \mu_i$, $\theta'_i = \lambda'_i - \mu'_i$

and $|\theta| = |\lambda| - |\mu|$,

a horizontal strip is a skew diagram with r squares which has at most one square in each column. Therefore for a horizontal strip

$$\theta'_i = \lambda'_i - \mu'_i = 0 \text{ or } 1 \text{ for each } i \ge 1.$$

Let $P_\lambda = P_\lambda(x;t)$ and $Q_\lambda = Q_\lambda(x;t)$ be Hall-Littlewood P and Q functions [Sultana 2001] respectively, such that

$$Q_\lambda = b_\lambda(t) P_\lambda \tag{1.1}$$

where $b_\lambda(t) = \prod_{i\ge 1} \phi_{m_i(\lambda)}(t)$, $m_i(\lambda)$ is multiplicity of i in λ.

$$\phi_m(t) = \prod_{i=1}^{m} (1 - t^i).$$

Then there is another function $q_r(x;t)$ given by

$$q_r(x;t) = Q_r(x;t) \tag{1.2}$$

$$= (1-t) \sum_{i=1}^{n} x_i^r \prod_{j\ne i} \frac{x_i - t.x_j}{x_i - x_j}$$

Special cases of Q_λ are S-functions when t = 0 and another class of symmetric functions, called Q-functions, when t = -1.

Rule for evaluating the product of two Schure-functions [Schur 1911] known as Littlewood-Richardson rule, was first stated but not proved by Littlewood and Richardson [1934]. The first proof of this rule is due to Las coux and Schutzenberger [1978] and Thomas [1974]. The method for the multiplication of Q-functions has been discussed by Morris [1962] which is similar to the method given by Murnaghan [1938].

THE PRODUCT OF TWO HALL-LITTLEWOOD FUNCTIONS

Macdonald [1995] has shown that the coefficient $f^{\upsilon}_{\lambda\mu}$(t) occurs in the product of two HL-functions are related to the Hall polynomials $g^{\upsilon}_{\lambda\mu}$. By using the Hall Algebra and horizontal strip, he has proved the following results.

Theorem

If μ is a partition of n and P_μ is HL-P functions on the partition μ, then

$$P_\mu P_{1^m} = \Sigma\, F^{\lambda}_{\mu 1^m}(t)\, P_\lambda.$$

where

$$F^{\lambda}_{\mu 1^m}(t) = \frac{V_\lambda(t)}{\prod V_{m_i - r_i}(t)\; V_{r_i}(t)}$$

And

$$V_r(t) = \prod_{j=1}^{r} \frac{(1-t^j)}{(1-t)}.$$

Theorem

If $\lambda \supset \mu$ and $\theta = \lambda - \mu$ is horizontal strip then

$$P_\mu P_r = \sum_{\lambda} F^{\lambda}_{\mu r}(t)\, P_\lambda(t)$$

where

$$F^{\lambda}_{\mu r}(t) = (1 - t)^{-1} \Big[\prod_{i \in I} (1 - t^{m_i(\lambda)}\Big]$$

where I is a set of integers i such that

$\theta'_i = 1$ and $\theta'_{i+1} = 0$ and $m_i(\lambda)$ is multiplicity of i in λ.

and $F^{\lambda}_{\mu r}(t) = 0$, otherwise.

Theorem

If $\lambda \supset \mu$ and $\theta = \lambda - \mu$ is a horizontal strip, then

$$Q_\mu q_r = \sum_{\lambda} f^{\lambda}_{\mu r}(t)\, Q_\lambda$$

where

$$f^{\lambda}_{\mu r}(t) = \sum_{\lambda} \frac{b_{\mu}(t)}{b_{\lambda}(t)} \prod_{i \in I} (1 - t^{m_i(\lambda)}).$$

Proof

By using the above Theorem and the definitions of Q_λ and q_r given in (1.1) and (1.2) we get the required result.

EXPLICIT FORMULA FOR $f^{\nu}_{\lambda(r-1,1)}$ (t)

Morris [1963] has given direct method for evaluating the product of two functions $Q_\lambda(x; t)$ and $Q_\nu(x; t)$.
That is, if we let

$$Q_\lambda Q_\mu = \Sigma f^{\upsilon}_{\lambda\mu}(t)\, Q_\nu$$

It was shown how to determine the coefficient $f^{\upsilon}_{\lambda\mu}(t)$. His method depends on the evaluation of the product

$$Q_\lambda(x; t)\, q_r(x; t) = \Sigma f^{\upsilon}_{\lambda\mu}(t)\, Q_\nu(x; t)$$

and a procedure was given by Morris [1964] for carrying out this multiplication. He proved that,

Theorem

If $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_{q1}^{m_q})$ is a partition of n, and

$$Q_\lambda(x; t)\, q_r(x; t) = \sum_{\upsilon} f^{\upsilon}_{\lambda\mu}(t)\, Q_\nu,$$

then $f^{\upsilon}_{\lambda\mu}(t) = 0$, if the Schur function does not appear in the product $S_\lambda h_r$ by means of Littlewood-Richardson rule, if S_ν does appear in the product $S_\lambda h_r$ and

$$\nu = (\lambda_1 + r_1,\ \lambda_1^{m_1 - 1},\ \lambda_2 + r_2,\ \lambda_2^{m_2 - 1}, \ldots, \lambda_q + r_q, \ldots, \lambda_{q1}^{m_q - 1}, r_{q+1})$$

where $\quad 0 \le r_i \le r \quad (i, = 1,\ldots,q + 1)$

$$\sum_{i=1}^{q+1} r_i = r,$$

then
$$f^{\upsilon}_{\lambda\mu}(t) = \prod_{i=1}^{q} Z_i(t) \tag{2.1}$$

where

$$Z_i(t) = \begin{cases} 1 - t^{m_i}, if\ r_i > 0 \ \ \& \ \ \lambda_{i+1} + r_{i+1} < \lambda_i \\ 1 \qquad , if\ r_i > 0 \ \ \& \ \ \lambda_{i+1} + r_{i+1} = \lambda_i \\ 1 \qquad , if\ r_i = 0 \end{cases}$$

On the basis of this theorem we shall prove the following result.

Theorem

If $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_{q1}^{m_q})$ is a partition of n such that $\lambda_1 > \lambda_2, \ldots > \lambda_q$

And $\quad Q_\lambda\, q_{r-1,1} = \sum_{\upsilon} f^{\nu}_{\lambda,(r-1,1)}$ (t) Q_ν,

then $f_{\lambda(r-1,1)} = 0$, if S_ν does not appear in the product $S_\lambda\, S_{r-1,1}$, and

$$f^{\nu}_{\lambda,(r-1,1)}\text{(t)} = \sum_{\mu} f^{\nu}_{\mu 1}\text{(t)}\ f^{\mu}_{\lambda,r-1}\text{(t)}$$

where $f^{\mu}_{\lambda,r-1}$(t) and $f^{\nu}_{\mu 1}$(t) are as defined in (2.1).

Proof

We know that

$$q_\lambda = q_{\lambda_1 \ldots \lambda_r} = q_{\lambda_1} q_{\lambda_2} \ \cdots\ q_{\lambda_r}.$$

Thus

$$\begin{aligned} Q_\lambda q_{r-1,1} &= Q_\lambda q_{r-1}\, q_1 \\ &= (Q_\lambda q_{r-1})\, q_1 \end{aligned}$$

Hence

$$= \Big(\sum_{\mu} f^{\mu}_{\lambda,r-1}\text{(t)}\ Q_\mu\Big)q_1$$

Again using above theorem we have

$$Q_\lambda q_{r-1,1} = \Big(\sum_{\mu} f^{\mu}_{\lambda,r-1}\text{(t)}\ \Big(\sum_{\nu} f^{\nu}_{\mu,1}\text{(t)}\ Q_\nu\text{(t)}\Big)$$

By substituting $\sum_{\mu} f^{\mu}_{\lambda,r-1}$(t) $f^{\nu}_{\mu,1}$(t) = $f^{\nu}_{\lambda,(r-1,1)}$(t)

we get the required result.

References

Lascoux, A. and Schutzenberger, M.P. (**1978**) "Sur une conjecture de H. O. Folkes" *C.R. Acad. Sci. Paris*, 286A, 323-324.

Littlewood, D.E. and Richardson, A.R. (**1934**) "Group characters and Algebra", *Phil. Trans. A*, 233, 99-141.

Macdonald, I.G. (**1995**) "Symmetric functions and Hall Polynomials", 2nd ed., Oxford Mathematical Monographs, Clarendon Press, Oxford.

Morris, A.O. (**1962**) "On Q-functions", *J. London Math. Soc.*, 37, 445-455.

Morris, A.O. (**1963**) "The multiplication of Hall functions", *Proc. London Math. Soc.,* 13, 733-742.

Morris, A.O. (**1964**) "A note on the multiplication of Hall functions", *J. London Math. Soc.,* 39, 481-488.

Murnaghan, F.D. (**1938**) "The theory of group representations", Baltimore.

Schur, I. (**1911**) "Uber die Derstellungs der Symmetrischen und der alternierenden Gruppe durch gebrochene Lineare Substitutionen", *J. reine angew. Math.,* 139, 155-250.

Sultana, N. (**2001**) "Some results on Hall-Littlewood Polynomials", *J. Natur. Sci. and Math.*, 41(1), 43-52.

Thomas, G.P. (**1974**) "Baxter Algebra and Schur functions", Ph.D. Thesis, University of Wales, U.K. Swansea.