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FIVE-FACTOR CENTRAL COMPOSITE DESIGNS ROBUST TO A PAIR OF MISSING OBSERVATIONS

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Abstract: Missing Observations in Central Composite Designs (c.c.d) are well investigated since the development of Minimaxloss criterion presented by Akhtar and Prescott [1986]. Here we investigate two missing values in different configurations of c.c.d with five factors and develop designs robust to a pair of missing observations under Minimaxloss criterion.

Keywords: Missing values, central composite designs, robust designs, minimaxloss criterion.

INTRODUCTION

Central Composite Design (c.c.d) is a combination of i) a factorial experiment of size 2^k or a fraction of it, ii) 2k axial points (Star points) two on each of k axes a distance $\pm\alpha$ from the center of the design and iii) one or more points at the centre of the design. Total design points n comprises of n_f factorial, n_a axial and n_c centre point.

For five factors the c.c.d consists of 32 factorial, 10 axial and one or more center points. Different configurations of five-factor c.c.d can be obtained by taking different replications of factorial and axial parts.

Central Composite Designs with different properties can be developed by taking different values of α, i.e. distance of axial points from the centre of the design. Box [1954] developed orthogonal c.c.d. Box and Hunter [1957] developed rotatable designs. Box and Draper [1959] discussed designs robust to inadequate model. Box and Draper [1975] studied robust to outliers, which are referred here as outlier robust designs. Designs robust to missing observations with different probability of missing for different observations are studied by Herzberg and Andrews [1975, 1976] and Andrews and Herzberg [1979].

Akhtar and Prescott [1986] studied the reduction in $|X'X|$ due to missing observations and developed minimaxloss criterion, which is actually minimizing the maximum loss due to missing observations. The loss is the relative reduction in $|X'X|$ due to missing observations. Akhtar and Prescott [1987] provide a comprehensive review of robust response surface designs.

Here we study two missing observations in a five-factor c.c.d with three different configurations of factorial (F) and axial (A) parts. These configurations are

(i) design with half replicate of factorial part (1/2 F+A)
(ii) design with one replication of factorial and axial part (F+A)
(iii) design with two replications of axial part (F+2A).

DIFFERENT GROUPS OF PAIRS WITH SIMILAR LOSSES

All possible pairs of observations ${}^nC_2$ can be placed in groups with similar losses. The main groups are ff, fa, aa, fc, ac and cc, where f, a and c are for a factorial, an axial and a centre point respectively.

Pairs of two factorial points, ff, can be further divided into groups, ff1, ff2, ff3, ff4, ff5, representing pairs of two factorial points having 1, 2, 3, 4 and 5 signs different. Examples for ff3 are

$$\begin{bmatrix} +1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 & -1 \end{bmatrix} \; or \; \begin{bmatrix} +1 & +1 & -1 & +1 & -1 \\ +1 & -1 & +1 & -1 & -1 \end{bmatrix}$$

Pairs of two axial points are of three types aa0, aa1 and aa2. aa0 are pairs of two axial observations at the same axial point and only occur when axial part is replicated twice. aa1 and aa2 are two axial points on the same axis and on different axes respectively. The examples of aa0, aa1 and aa2 are

$$\begin{bmatrix} +\alpha & 0 & 0 & 0 & 0 \\ +\alpha & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & +\alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 \end{bmatrix} \; and \; \begin{bmatrix} 0 & +\alpha & 0 & 0 & 0 \\ 0 & 0 & +\alpha & 0 & 0 \end{bmatrix} \text{respectively.}$$

Pairs of a factorial and an axial observation fa can be further divided as fa1 and fa2. fa1 are pairs of factorial and axial observations on the same side of the cube such as

$$\begin{bmatrix} 1 & -1 & -1 & +1 & +1 \\ +\alpha & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 & +1 & +1 \\ 0 & -\alpha & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & +1 & +1 & +1 & -1 \\ 0 & 0 & +\alpha & 0 & 0 \end{bmatrix}$$

(α and corresponding 1 have same signs)

and fa2 are pairs in which factorial and axial points are on opposite sides of the cube such as

$$\begin{bmatrix} 1 & -1 & -1 & +1 & +1 \\ -\alpha & 0 & 0 & 0 & 0 \end{bmatrix} \; and \; \begin{bmatrix} 1 & -1 & -1 & +1 & +1 \\ 0 & 0 & +\alpha & 0 & 0 \end{bmatrix}$$

(α and corresponding 1 have opposite signs).

MINIMAXLOSS CRITERION

Akhtar and Prescott [1986] developed minimaxloss criterion to find designs robust to missing observations.

Let the postulated underlying model is

$$\underline{y} = X\,\underline{\beta} + \underline{\varepsilon}$$

where $\underline{y}$ is a vector of responses at n design points, X is a matrix of order $n\times p$ formed from the p input variables in the response model, $\underline{\beta}$ is a vector of p coefficients and $\underline{\varepsilon}$ is an error vector of order n.

The usual least square estimates are

$$\underline{\hat{\beta}} = (X'X)^{-1} X'\underline{y}$$

$$\text{Var}(\underline{\hat{\beta}}) = (X'X)^{-1} \sigma^2$$

$$\underline{\hat{y}} = X(X'X)^{-1} X'.\underline{y} = R.\underline{y} \text{ and}$$

$$\text{Var}(\underline{\hat{y}}) = [X(X'X)^{-1} X'] \sigma^2 = R\sigma^2$$

where R is a matrix of order n and is sometimes called hat matrix.
Under D-optimality we maximize $|X'X|$. For c.c.d $|X'X|$ is an increasing function of α and is maximum at $\alpha = \infty$.
If there are two missing observations i and j, the $|X'X|$ for the complete design is reduced to $_{ij}|X'X|$. We want this reduction to be as small as possible. After some algebra it can be shown that

$$_{ij}|X'X| = |X'X| A_{ij}$$

where A_{ij} is the diagonal values of the second compound of (I - R) matrix. For compounds of matrix see Aitken and Rutherford [1964].
Akhtar and Prescott [1986] defined 'loss' as the relative reduction in $|X'X|$. Loss due to a pair of missing observations (i, j)

$$L_{ij} = \frac{|X'X| - {}_{ij}|X'X|}{|X'X|}$$

$$= 1 - A_{ij}$$

$$= 1 - (1 - r_{ii})(1 - r_{jj}) - r_{ij}^2$$

where r_{ii}, r_{jj} and r_{ij} are corresponding elements of R. For particular n and p, ΣA_{ij} is constant which means that ΣL_{ij} is also constant which implies that a particular loss L_{ij} can be reduced only at the cost of increasing other losses. All losses being equal will be ideal.
The most useful criterion for reducing the losses will be to minimize the maximum loss due to a pair of missing observations, which is called minimaxloss criterion.

FIVE-FACTOR DEISGNS WITH HALF FACTORIAL REPLICATE

Five-factor c.c.d consisting of $n_f = 16$ points from half replicate of factorial part with highest order interaction as defining contrast, $n_a = 10$ points of axial part and three or more centre points. Total design points are 29 or more. The loss due to different pairs of missing points is studied over a range of α from 1.0 to 4.0. Among the possible groups of pairs, ff1, ff3 and ff5 are empty. Only ff2, ff4, aa1, aa2, fa1, fa2, fc, ac and cc are non-empty.
The computations of losses have shown that

$$L_{ff4} > L_{ff2}; \quad L_{aa1} > L_{aa2} \text{ and } L_{fa1} > L_{fa2}.$$

The losses due to fc, ac and cc remain less for the whole range of α as long as the number of center points is three or more. The losses due to ff4, aa1 and fa1 are plotted against α in Fig. 1.

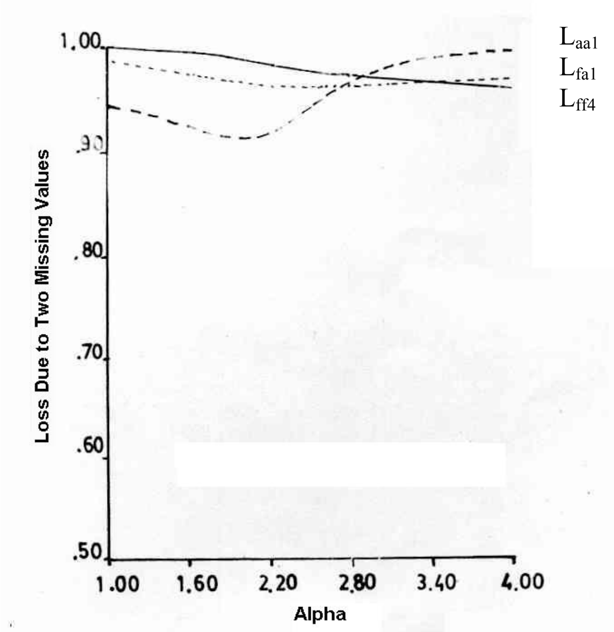

Fig. 1: Losses due to a pair of missing observations for design with k = 5, n_f = 16, n_a = 10 and n_c = 3 plotted against α.

Table 1: Loss due to different pairs of missing observations
No. of variables k = 5 Total design points n = 29
No. of parameters p = 21 No. of centre points = 3
{Half replicate of factorial part]

Design	Alpha	$\lvert X'X \rvert$ for complete design	Maxloss due to two factorial observation missing	Maxloss due to two axial observation missing	Maxloss due to one axial and one factorial observation missing	Overall maxloss due to a pair of observation missing	Maxloss due to a single observation missing
Alpha = 1.0	1.0000	0.2520E+23	0.9992	0.9437	0.9868	0.9992	0.9648
Orthogonal	1.6644	0.1260E+27	0.9942	0.9199	0.9723	0.9942	0.9132
Rotatable	2.0000	0.3378E+28	0.9876	0.9130	0.9654	0.9876	0.8795
Alpha = $\sqrt{k}$	2.2361	0.3185E+29	0.9815	0.9243	0.9622	0.9815	0.8558
Outlier robust	2.4482	0.2451E+30	0.9769	0.9437	0.9614	0.9769	0.8385
Minimaxloss2	2.7929	0.6504E+31	0.9716	0.9716	0.9625	0.9716**	0.8191
Minimaxloss1	3.4972	0.2684E+34	0.9651	0.9932	0.9668	0.9932	0.7937

* Minimaxloss due to one missing observation. ** Minimaxloss due to two missing observations.

The loss L_{ff4} which is near 1 for α = 1, decreases with the increase of α, L_{fa1} first decreases and then increases slowly with increasing α. L_{aa1} which, for α = 1, is less than first two losses, decreases up to α = 2.0 and then increases with increasing α. Maximumloss which corresponds to L_{ff4} and then to L_{aa1} is minimum when $L_{ff4} = L_{aa1}$ at α = 2.7929. Thus the five-factor c.c.d of this configuration with n_f = 16, n_a = 10, n_c = 3, and α = 2.7929 is a minimaxloss design robust to a pair of missing observations.

The losses L_{ff4}, L_{aa1}, L_{fa1}, maximumloss, $|X'X|$ and maximumloss due to single missing observation for different designs of the same configuration are shown in Table 1.

From now on minimaxloss1 and minimaxloss2 designs mean minimaxloss designs robust to one and two missing observations respectively.

For robustness to a pair of missing observations, outlier robust design with α = 2.4482 is second best after minimaxloss2 design. When there is a single missing observation the minimaxloss2 design performs better than all designs other than minimaxloss1 design. The latter does not perform well when two observations are missing.

Table 2: Variances of parameter estimates for complete design and for designs with a pair of observation missing.
No. of variables k = 5 Total design points n = 29
No. of parameter p = 21 No. of centre points = 3
{Half replicate of factorial part]

Alpha	n	Variances of Parameters Estimates						Sum of Variance
		Inter-cept	Linear (min)	Linear (max)	Quadratic (max)	Quadratic (max)	Inter-action	
1.0000	29	0.1082	0.0556	0.0556	0.4077	0.4077	0.0625	3.0492
	27ff	0.1232	0.1667	0.4704	0.4360	0.4360	0.5876	7.7781
	27aa	0.3333	0.0556	0.0625	0.8333	5.0625	0.0625	5.6254
	27fa	0.1232	0.1667	0.2027	0.4360	0.8578	0.2031	5.6254
1.6644	29	0.2142	0.0464	0.0464	0.0652	0.0652	0.0625	1.3971
	27ff	0.2484	0.0799	0.1420	0.0776	0.0776	0.2358	3.0060
	27aa	0.3333	0.0464	0.0625	0.1086	0.3888	0.0625	2.0297
	27fa	0.2307	0.0795	0.1230	0.0712	0.1394	0.1224	2.3199
2.0000	29	0.3043	0.0417	0.0417	0.0408	0.0408	0.0625	1.3415
	27ff	0.3171	0.0625	0.0884	0.0457	0.0457	0.1677	2.3277
	27aa	0.3333	0.0417	0.0625	0.0521	0.1875	0.0625	1.5833
	27fa	0.3100	0.0617	0.1092	0.0430	0.0852	0.1075	1.9984
2.2361	29	0.3333	0.0538	0.0385	0.0309	0.0309	0.0625	1.3050
	27ff	0.3333	0.0538	0.0692	0.0321	0.0321	0.1437	2.0534
	27aa	0.3333	0.0853	0.0625	0.0333	0.1558	0.0625	1.4638
	27fa	0.333	0.0529	0.1030	0.0314	0.0653	0.1007	1.8458
2.4482	29	0.3117	0.0357	0.0357	0.0230	0.0230	0.0625	1.2306
	27ff	0.3227	0.0477	0.0477	0.0231	0.0231	0.1343	1.8897
	27aa	0.3333	0.0357	0.0357	0.0232	0.1551	0.0625	1.4116
	27fa	0.3162	0.0468	0.0468	0.0231	0.0513	0.0964	1.7096
2.7929	29	0.2356	0.0316	0.0316	0.0133	0.0133	0.0625	1.0856
	27ff	0.2809	0.0398	0.0491	0.0135	0.0135	0.1304	1.7157
	27aa	0.3333	0.0316	0.0625	0.0137	0.1745	0.0625	1.3767
	27fa	0.2519	0.0392	0.0941	0.0134	0.0340	0.0918	1.5083
3.4972	29	0.1349	0.0247	0.0247	0.0047	0.0047	0.0625	0.9072
	27ff	0.2035	0.0288	0.0353	0.0050	0.0050	0.1301	1.5132
	27aa	0.3333	0.0247	0.0625	0.0056	0.2268	0.0625	1.3688
	27fa	0.1547	0.0285	0.0883	0.0048	0.0157	0.0868	1.2598

ff- A pair of factorial observations on far corners of cube missing.
aa- A pair of axial observations on the same axis missing.
fa- A factorial observation and a nearest axial observation missing.

The variances of the parameter estimates for the designs under discussion and corresponding to missing pairs from groups ff4, fa1 and aa1, are shown in Table 2. The variances for intercept are 1/3 or less. The increase in the variance of the quadratic parameter estimate corresponding to missing pairs of two axial points is comparatively large. The increase in the variance of the interaction parameter estimate due to a missing pair of two factorial points is also large. The variances of the linear, quadratic and interaction parameter estimates for complete or reduced minimaxloss2 designs are small.

FIVE-FACTOR DESIGNS WITH SINGLE REPLICATE OF FACTORIAL AND AXIAL PARTS

Five-factor c.c.d with one replication of factorial and axial parts have n_f = 32, n_a =10 and three or more center points. There are 45 or more design points. From the possible groups of pairs of missing observations, aa0 is empty.

The losses due to different pairs of missing observations have been studied for a range of α from 1.0 to 3.0. The computations of losses have shown that

$$L_{ff5} > L_{ff1} > L_{ff3} > L_{ff4} > L_{ff2},$$
$$L_{aa1} > L_{aa2}$$
and $$L_{fa1} > L_{fa2}.$$

The losses L_{ff5}, L_{fa1}, and L_{aa1} are plotted against α in Fig. 2. It is observed from this figure that L_{ff5} decreases and L_{fa1} increases with increasing α. L_{aa1} decreases and increases with increase in α and is minimum for α=2.0865. Thus the five-factor design of this configuration with α = 2.0865 is a minimaxloss2 design robust to two missing observations.

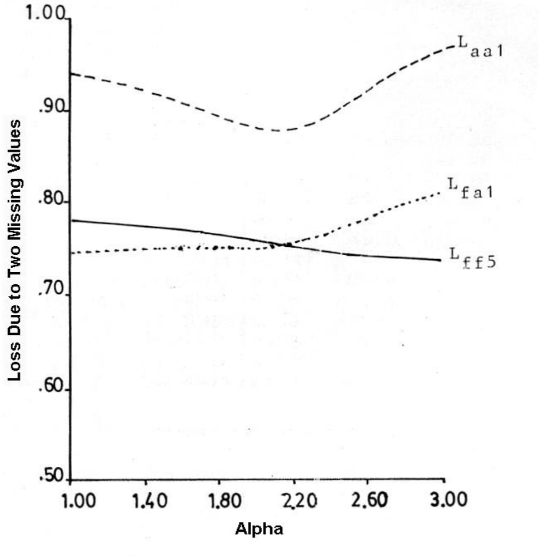

Fig. 2: Losses due to a pair of missing observations for design with k = 5, n_f = 32, n_a = 10 and n_c = 3 plotted against α.

Table 3: Loss due to different pairs of missing observations
No. of variables k = 5 Total design points n = 45
No. of parameters p = 21 No. of centre points = 3

Design	Alpha	$\lvert X'X \rvert$ for complete design	Maxloss due to two fact. obs. missing	Maxloss due to two axial obs. missing	Maxloss due to one axial and one fact obs. missing	Overall maxloss due to a pair of obs. missing	Maxloss due to a single obs. missing
Alpha = 1.0	1.0000	0.1236E+28	0.7792	0.9402	0.7451	0.9402	0.4976
Orthogonal	1.7244	0.6895E+31	0.7668	0.8983	0.7494	0.8983	0.5180
Minimaxloss2	2.0865	0.1612E+33	0.7556	0.8784	0.7516	0.8784**	0.5296
Outlier robust	2.1265	0.2265E+33	0.7542	0.8788	0.7527	0.8788	0.5324
Alpha = $\sqrt{k}$	2.2361	0.5795E+33	0.7506	0.8839	0.7570	0.8839	0.5429
Rotatable	2.3784	0.2014E+34	0.7465	0.8986	0.7656	0.8986	0.5620
Minimaxloss1	0.7045	0.4614E+25	0.7822	0.9477	0.7421	0.9477	0.4953*

* Minimaxloss due to one missing observation. ** Minimaxloss due to two missing observations

The losses L_{ff5}, L_{fa1}, L_{aa1} and maximum losses due to one or two missing observations for different c.c.d of the same configuration are shown in Table 3. The maximumloss due to a pair of missing observations for minimumloss2 design is very near to outlier robust design because the α values for both designs are close. The maximum losses for other designs are higher. Minimaxloss1 design is in the middle of the designs discussed here.
The variances of the parameter estimates for the designs under discussion and corresponding to pairs from ff5, fa1 and aa1 missing are shown in the Table 4. The variance of the linear and the interaction parameter estimates are small for all complete and reduced designs. The variances of the quadratic parameter estimates are also small for complete designs with $\alpha > 1$. The increases due to a pair of missing observations in these variances are also small except the increase in one quadratic parameter estimate due to a pair of axial observations missing.

FIVE-FACTOR DESIGNS WITH AXIAL PART REPLICATED TWICE

Five-factor designs with complete factorial replicate and two replications of the axial part have $n_f = 32$, $n_a = 20$ and $n_c = 3$ or more. There are 55 or more design points.
The losses due to different pairs of missing observations have been studied for three center points and over a range of α from 1.0 to 4.0. The computations of losses have shown that for the whole range of α studied

$$L_{ff5} > L_{ff1} > L_{ff3} > L_{ff4} > L_{ff2},$$
$$L_{aa0} > L_{aa1} > L_{aa2}$$

and

$$L_{fa1} > L_{fa2}.$$

The maximumloss in minimum for $\alpha = 2.7547$. Thus the five-factor design of this configuration with $\alpha = 2.7547$ is a minimaxloss2 design robust to a pair of missing observations.

Table 4: Variances of parameter estimates for complete design and for designs with a pair of observation missing.

No. of variables k = 5 Total design points n = 45

No. of parameters p = 21 No. of centre points = 3

Alpha	n	Variances of Parameters estimates						Sum of Variance
		Inter-cept.	Linear (min)	Linear (max)	Quadratic (max)	Quadratic (max)	Inter-action	
1.0000	45	0.1073	0.0294	0.0294	0.4060	0.4060	0.0313	2.5966
	43ff	0.1075	0.0319	0.0319	0.4063	0.4063	0.0375	2.6732
	43aa	0.3333	0.0294	0.0313	0.8333	5.0312	0.0313	9.1593
	43fa	0.1216	0.0311	0.0331	0.4330	0.7006	0.0332	3.0434
1.7244	45	0.2223	0.0264	0.0264	0.0565	0.0565	0.0313	0.9503
	43ff	0.2247	0.0282	0.0282	0.0570	0.0570	0.0374	1.0253
	43aa	0.3333	0.0264	0.0313	0.0942	0.2972	0.0313	1.4566
	43fa	0.2371	0.0277	0.0326	0.0613	0.0877	0.0331	1.0445
2.0865	45	0.3190	0.0246	0.0246	0.0356	0.0356	0.0313	0.9323
	43ff	0.3195	0.0262	0.0262	0.0359	0.0359	0.0373	1.0026
	43aa	0.3333	0.0246	0.0313	0.0440	0.1390	0.0313	1.0902
	43fa	0.3214	0.0257	0.0321	0.0370	0.0534	0.0331	0.9888
2.1265	45	0.3254	0.0244	0.0244	0.0341	0.0341	0.0313	0.9305
	43ff	0.3257	0.0259	0.0259	0.0344	0.0344	0.0373	1.0000
	43aa	0.3333	0.0244	0.0313	0.0408	0.1335	0.0313	1.0710
	43fa	0.3268	0.0255	0.0321	0.0353	0.0514	0.0331	0.9842
2.2361	45	0.3333	0.0238	0.0238	0.0303	0.0303	0.0313	0.9163
	43ff	0.3333	0.0253	0.0253	0.0304	0.0304	0.0372	0.9841
	43aa	0.3333	0.0238	0.0313	0.0333	0.1246	0.0313	1.0302
	43fa	0.3333	0.0249	0.0320	0.0308	0.0467	0.0331	0.9656
2.3784	45	0.3202	0.0231	0.0231	0.0254	0.0254	0.0313	0.8749
	43ff	0.3209	0.0245	0.0245	0.0254	0.0254	0.1372	0.9420
	43aa	0.3333	0.0231	0.0313	0.0260	0.1223	0.0313	0.9959
	43fa	0.3223	0.0241	0.0320	0.0255	0.0411	0.0331	0.9244
0.7045	45	0.0902	0.0303	0.0303	1.6289	1.6289	0.0313	8.6986
	43ff	0.0902	0.0329	0.0329	1.6292	1.6292	0.0375	8.7758
	43aa	0.3333	0.0303	0.0313	3.3829	24.7616	0.0313	39.0917
	43fa	0.1030	0.0321	0.0331	1.7213	2.8510	0.0332	10.3328

ff- A pair of factorial observations on far corners of cube missing.

aa- A pair of axial observations on the same axis missing.

fa- A factorial observation and a nearest axial observation missing.

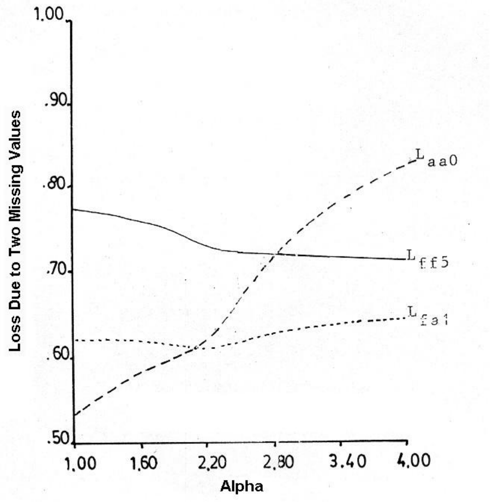

Fig. 3: Losses due to a pair of missing observations for design with k = 5, n_f = 32, n_a = 20 and n_c = 3 plotted against α.

Table 5: Loss due to different pairs of missing observations
No. of variables k = 5 Total design points n = 55
No. of parameters p = 21 No. of centre points = 3
{Axial observations replicated twice.}

Design	Alpha	$\lvert X'X \rvert$ for complete design	Maxloss due to two fact. obs. missing	Maxloss due to two axial obs. missing	Maxloss due to one axial and one fact obs. missing	Overall maxloss due to a pair of obs. Missing	Maxloss due to a single obs. missing
Alpha = 1.0	1.0000	0.4427E+29	0.7741	0.5367	0.6222	0.7741	0.4976
Orthogonal	1.5774	0.7495E+32	0.7594	0.5834	0.6202	0.7594	0.5180
Rotatable	2.0000	0.3850E+34	0.7407	0.6067	0.6134	0.7407	0.5296
Alpha = $\sqrt{k}$	2.2361	0.3340E+35	0.7282	0.6308	0.6122	0.7282	0.5324
Outlier robust	2.4978	0.4536E+36	0.7213	0.6771	0.6192	0.7213	0.5429
Minimaxloss2	2.7547	0.6059E+37	0.7197	0.7197	0.6275	0.7197**	0.5620
Minimaxloss1	3.5293	0.5793E+40	0.7149	0.7994	0.6413	0.7994	0.4953*

* Minimaxloss due to one missing observation.
** Minimaxloss due to two missing observations.

Table 6: Variances of parameter estimates for complete design and for designs with a pair of observation missing.
No. of variables k = 5 Total design points n = 55
No. of parameters p = 2 No. of centre points = 3
{axial part replicated twice 3}

Alpha	n	Variances of Parameters estimates						Sum of Variances
		Inter-cept.	Linear (min)	Linear (max)	Quadratic (max)	Quadratic (max)	Inter-action	
1.0000	55	0.0646	0.0278	0.0278	0.2043	0.2043	0.0313	1.5376
	53ff	0.0648	0.0299	0.0299	0.2047	0.2047	0.0375	1.6128
	53aa	0.0714	0.0278	0.0288	0.2140	0.2591	0.0313	1.6850
	53fa	0.0681	0.0293	0.0304	0.2093	0.2542	0.033`	1.6383
1.5774	55	0.1357	0.0238	0.0238	0.0404	0.0404	0.0313	0.7693
	53ff	0.1378	0.0253	0.0253	0.0410	0.0410	0.0374	0.8436
	53aa	0.1453	0.0238	0.0258	0.0430	0.0487	0.0313	0.8074
	53fa	0.1410	0.0249	0.0271	0.0418	0.0476	0.0331	0.8133
2.0000	55	0.2800	0.0208	0.0208	0.0262	0.0263	0.0313	0.8279
	53ff	0.2824	0.0219	0.0219	0.0267	0.0267	0.0372	0.8978
	53aa	0.2844	0.0208	0.0233	0.0271	0.0297	0.0313	0.8467
	53fa	0.2844	0.0216	0.0244	0.0268	0.0294	0.0330	0.8603
2.2361	55	0.3333	0.0192	0.0192	0.0221	0.0221	0.0313	0.8525
	53ff	0.3333	0.0201	0.0201	0.0222	0.0222	0.0371	0.9155
	53aa	0.3333	0.0192	0.0219	0.0222	0.0250	0.0313	0.8640
	53fa	0.3333	0.0199	0.0229	0.0222	0.0250	0.0330	0.8792
2.4978	55	0.2761	0.0176	0.0176	0.0148	0.0148	0.0313	0.7501
	53ff	0.2804	0.0183	0.0183	0.0148	0.0148	0.0370	0.8161
	53aa	0.2796	0.0176	0.0205	0.0148	0.0175	0.0313	0.7649
	53fa	0.2789	0.0181	0.0213	0.0148	0.0175	0.0329	0.7785
2.7547	55	0.1924	0.0160	0.0160	0.0089	0.0089	0.0313	0.6296
	53ff	0.2014	0.0167	0.0165	0.0090	0.0090	0.0371	0.7005
	53aa	0.1975	0.0160	0.0191	0.0089	0.0110	0.0313	0.6452
	53fa	0.1972	0.0165	0.0198	0.0089	0.0110	0.0329	0.6589
0.5293	55	0.0825	0.0122	0.0122	0.0025	0.0025	0.0313	0.4684
	53ff	0.0920	0.0126	0.0126	0.0025	0.0025	0.0373	0.5400
	53aa	0.0846	0.0122	0.0153	0.0025	0.0033	0.0313	0.4784
	53fa	0.0859	0.0125	0.0157	0.0025	0.0033	0.0329	0.4935

ff- A pair of factorial observations on far corners of cube missing.
aa- A pair of axial observations on the same axial missing.
fa- A factorial observation and a nearest axial observation missing.

The losses L_{ff5}, L_{aa0}, L_{fa1} are plotted against α in Fig. 3. The losses L_{ff5}, L_{aa0}, L_{fa1} and the maximum losses due to one or two missing observations for minimaxloss2 and other designs of the same configuration are shown in Table 5. The maximum loss due to a missing pair of observations in minimaxloss2 design is 0.7197 as compared to 0.7213 for outlier robust design. The maximum losses for other designs are larger. When there is only one missing observation the minimaxloss2 design performs better than all other designs except minimaxloss1 design.

The variances of the parameter estimates for the designs under discussion and corresponding to a missing pair of observations from groups ff5, aa0 and fa1 are shown in Table 6. Var (β_o) is around 0.2 for minimaxloss2 design. The variance of the linear parameter estimates for all complete and reduced designs are small.

The variance of the quadratic parameter estimates are also small for all complete and reduced designs with α > 1. For designs with α = 1 these variances are around 0.25. It is interesting to note that all the increases in the variances due to different missing pairs of observations are small. All the variances are very small for complete and reduced minimaxloss1 and minimaxloss2 designs.

CONCLUSIONS

Five-factor c.c.d robust to a pair of missing observations, for different configuration are given in Table 7. The losses due to two and one missing observation given in the last two columns of Table 7 are minimaxlosses for that configuration. If there is risk of a pair of observations missing in the experiment then the minimaxloss2 designs developed for each configuration is recommended. The configuration with factorial part replicated twice has not been discussed as the α for minimaxloss2 design in that case is less than one, which means the axial points are inside the cube of factorial points. This design also loses its natural balance of factorial and axial points in favour of factorial points.

The problem can be investigated further for more factors and larger number of missing observations.

Table 7: Five-Factor c.c.d robust to a pair of missing observations.

n_f	n_a	n_c	N	α	$\lvert X'X \rvert$	Minimaxloss2	Minimaxloss1
16	10	3	29	2.7929	0.6504E+31	0.9716	0.8191
32	10	3	45	0.0865	0.1612E+33	0.8784	0.5296
32	20	3	55	2.7547	0.6059E+37	0.7197	0.4133

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